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Risk and Uncertainty in Engineering Computations
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Interval Analysis Using Multilevel Quasi-Monte Carlo

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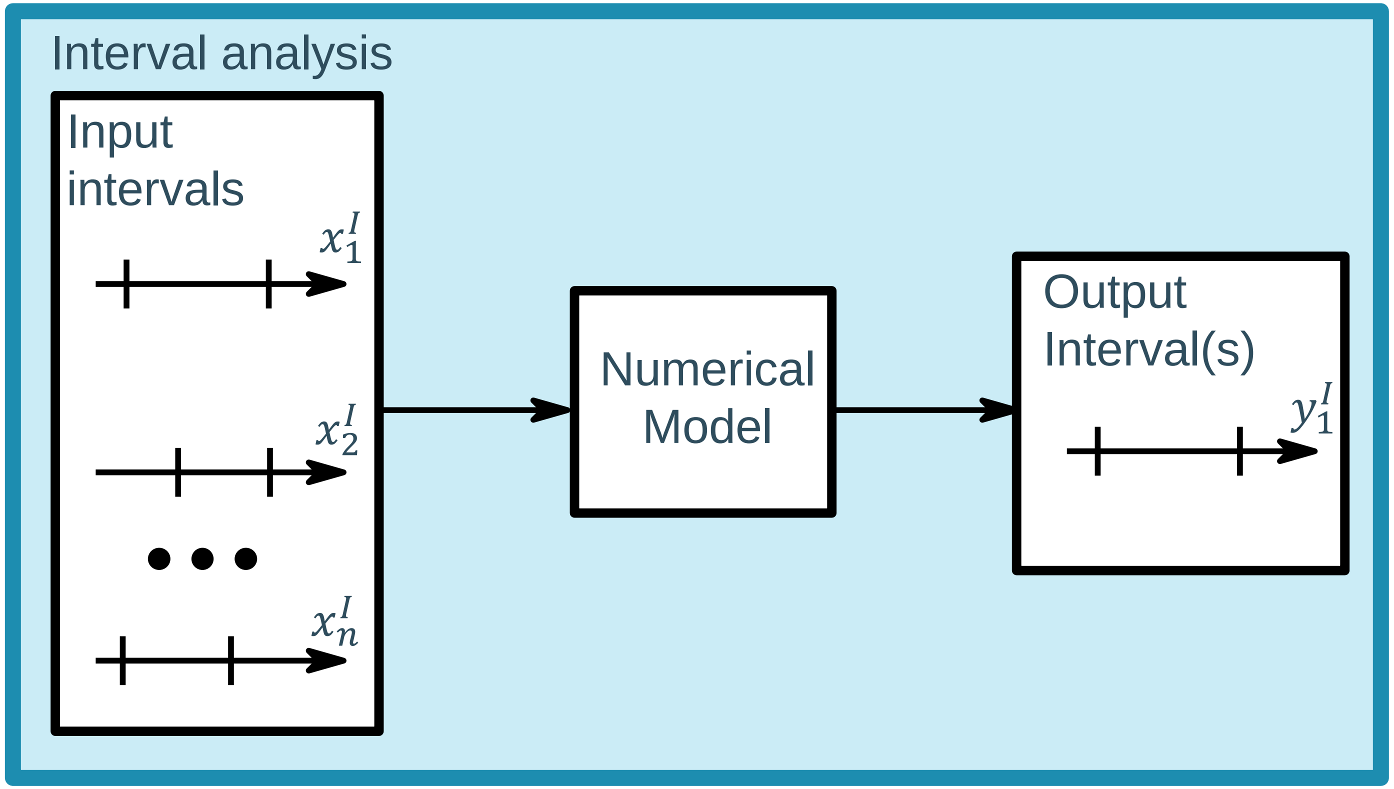
RELIABLE & ROBUST DESIGN

Introduction

- Global optimisation
  - Max and min the model respons
  - Optimisation procedure for each output bound
- Vertex analysis
  - For monotonic models
  - Propagate set of vertex points of $\boldsymbol{x}^I$
  - 2^n model evaluations
  - n=10: 1 024
  - n=20: 1 048 576

Problematic for:

- High number of input intervals
- High dimensional models

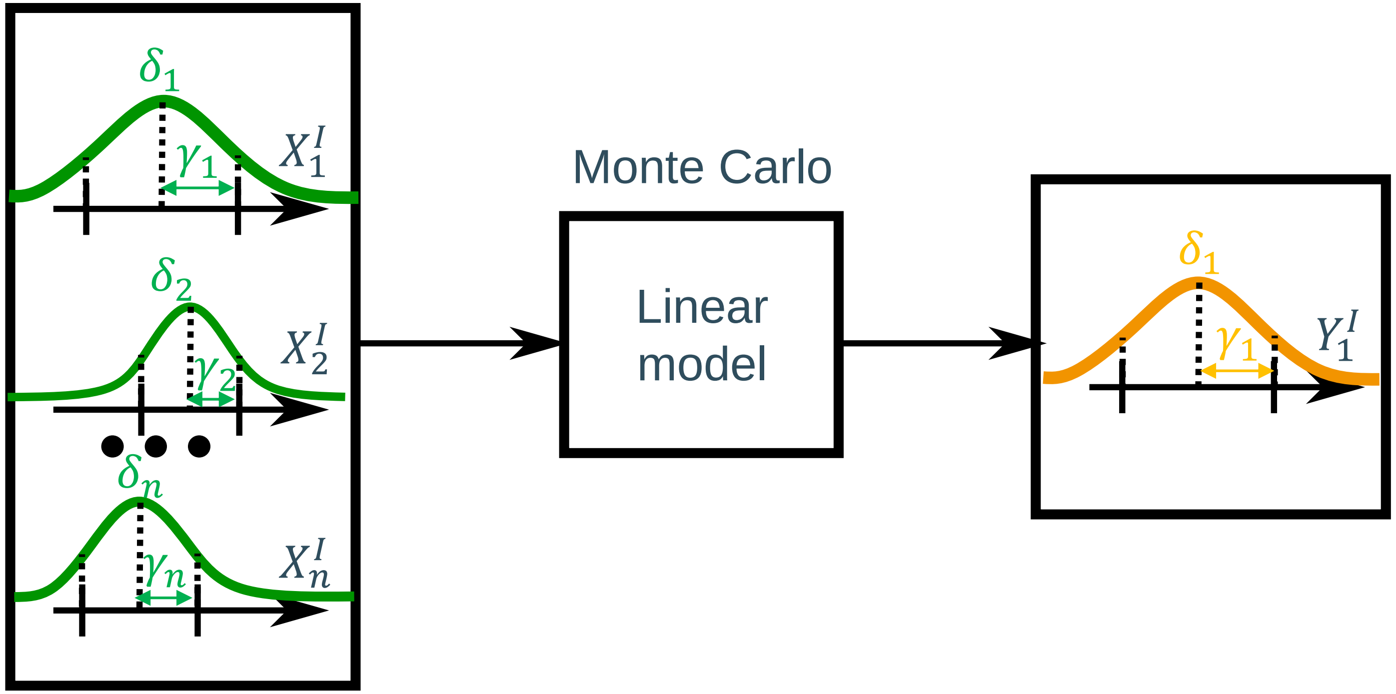


Overview

- Introduction
- **Quasi-Monte Carlo for interval analysis**
- Multilevel Monte Carlo
- Multilevel Quasi-Monte Carlo for interval analysis
- Numerical example
- Conclusions

Cauchy distributions to represent intervals

- Intervals as Cauchy variables
  - location $\delta_1 = x_{1m} = \frac{\overline{x_1}+\underline{x_1}}{2}$
  - shape parameter $\gamma_1 = x_{1w} = \frac{\overline{x_1}-\underline{x_1}}{2}$
- A linear combination of X is also Cauchy distributed with parameters
  $\delta_1 = c_1\delta_1 + c_2\delta_2 + \cdots + c_n\delta_n$
  $\gamma_1 = c_1\gamma_1 + c_2\gamma_2 + \cdots + c_n\gamma_n$
- Model: $c_1, c_2, \ldots, c_n$
- For linear models this is also the output interval
- $X_1, \ldots, X_n$ independent random variables
- Sampling techniques for Cauchy variables can be used



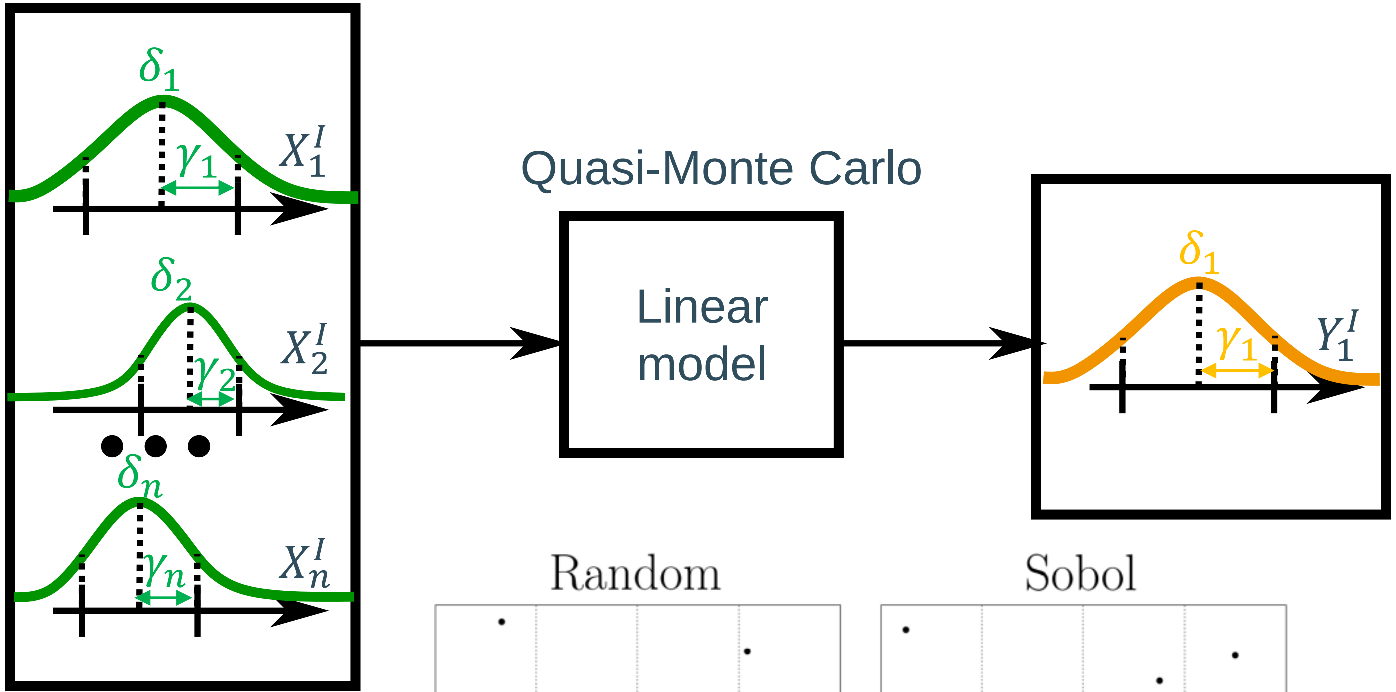
Vladik Kreinovich and Scott A. Ferson. A new cauchy-based black-box technique for uncertainty in risk analysis. *Reliability Engineering & System Safety*, 85(1-3):267{279, 2004.

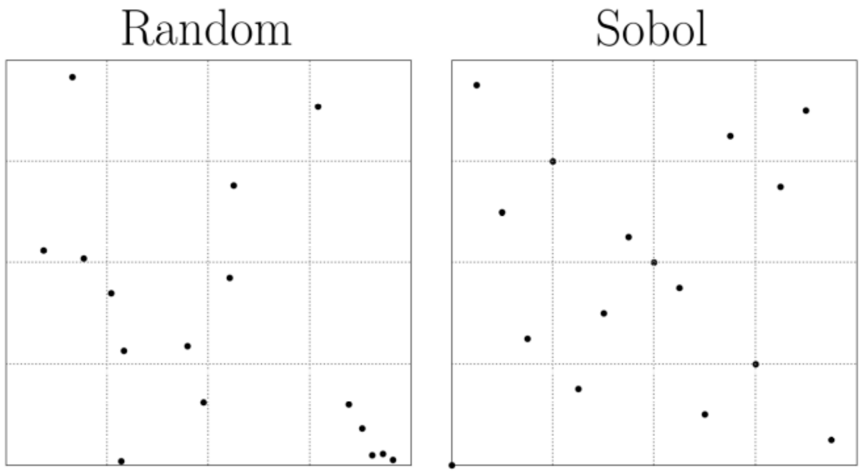
Quasi-Monte Carlo for Interval analysis

- Sampling from Cauchy variable

$$F_{X_1}^{-1}(\delta_1, \gamma_1) = \delta_1 + \gamma_1 \tan\left[\pi\left(p - \frac{1}{2}\right)\right]$$

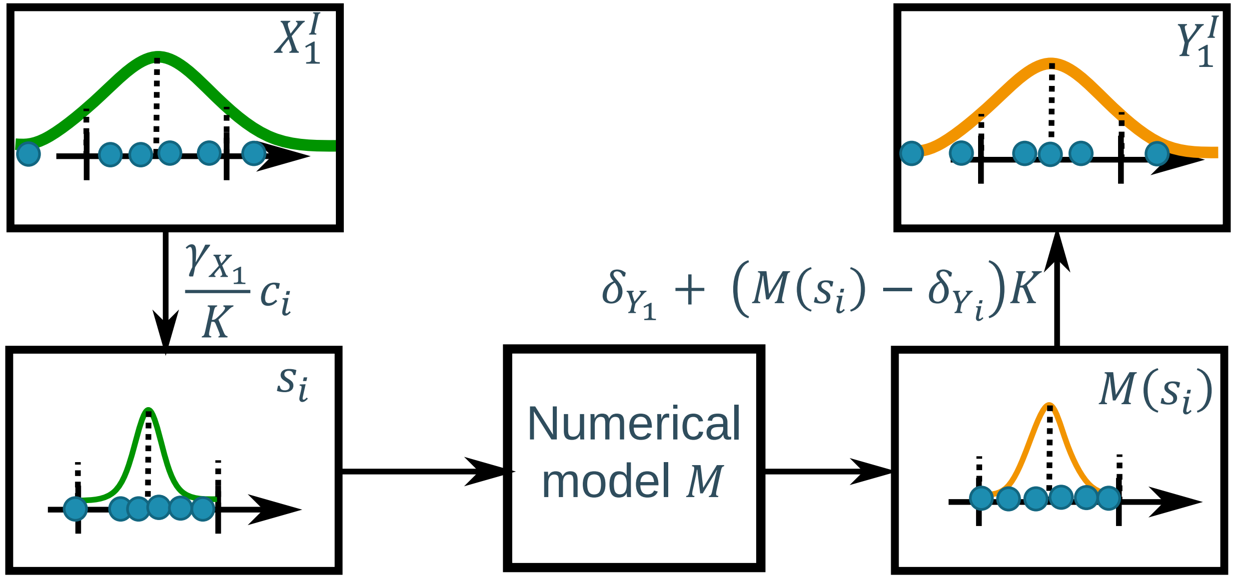
- Sampling from $p \in [0,1]$
- Quasi-Monte Carlo sampling techniques can be used
  - Sobol sequence
  - Latin hypercube
- Estimating output parameters
  - δ_1 median as mean is undefined
  - γ_1 Hodges-Lehmann Estimator or Maximum Likelihood Estimator





Quasi-Monte Carlo for Interval analysis

- Sampling within interval bounds
  - Cauchy support $(-\infty, +\infty)$
  - Model can be nonlinear outside interval bounds
  - Unphysical values
- Input: scale samples to fit interval bounds
  - Take c_i samples from $C(0,1)$
  - Find $K = \max(|c_i|)$
  - Scale: $s_i = \delta_{X_1} + \left(\frac{\gamma_{X_1}}{K} c_i\right)$
- Output: rescale samples to output distribution
  - Find median δ_{Y_1}
  - Rescale: $y_{Y_1} = \delta_{Y_1} + \left(M(s_i) - \delta_{Y_i}\right)K$



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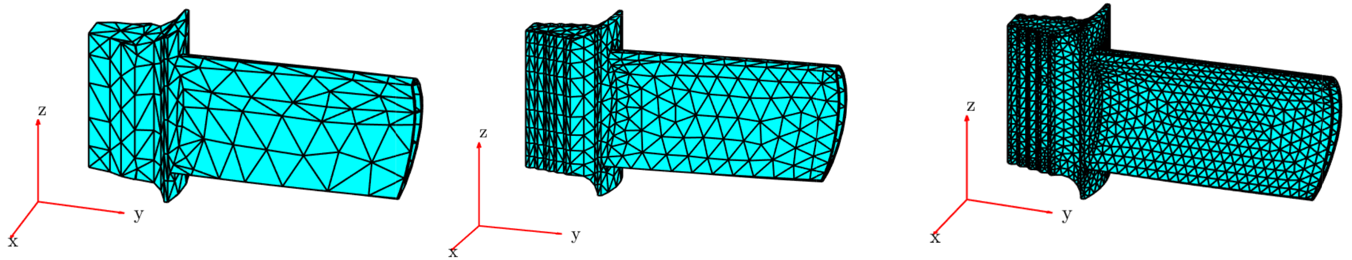
Multilevel Monte Carlo

- Discretised model
  - Time step size
  - Finite element size
  - Basis function order
- Multilevel
  - Take many samples on low level
  - Accurate estimate of low level
  - Correct this estimate to reduce bias
  - Correction estimate has low variance
  - Strong correlation between level l and $l-1$

$$h^l = 2^{l-1} h^{l=0}$$

L=0 L=1 L=2



Fast with high bias

Slow with low bias

MLMC Estimator | Estimator on a cheap level | Estimator of a correction term

$$E(y)_L = \mathbb{E}(y)_{l=0} + \sum_{l=1}^{L} \mathbb{E}(y_l - y_{l-1})$$

Pieterjan Robbe, Multilevel Uncertainty Quantification Methods for Robust Design of Industrial Applications, PhD Thesis, KU Leuven, 2019.

Multilevel Monte Carlo

- Mean square error:

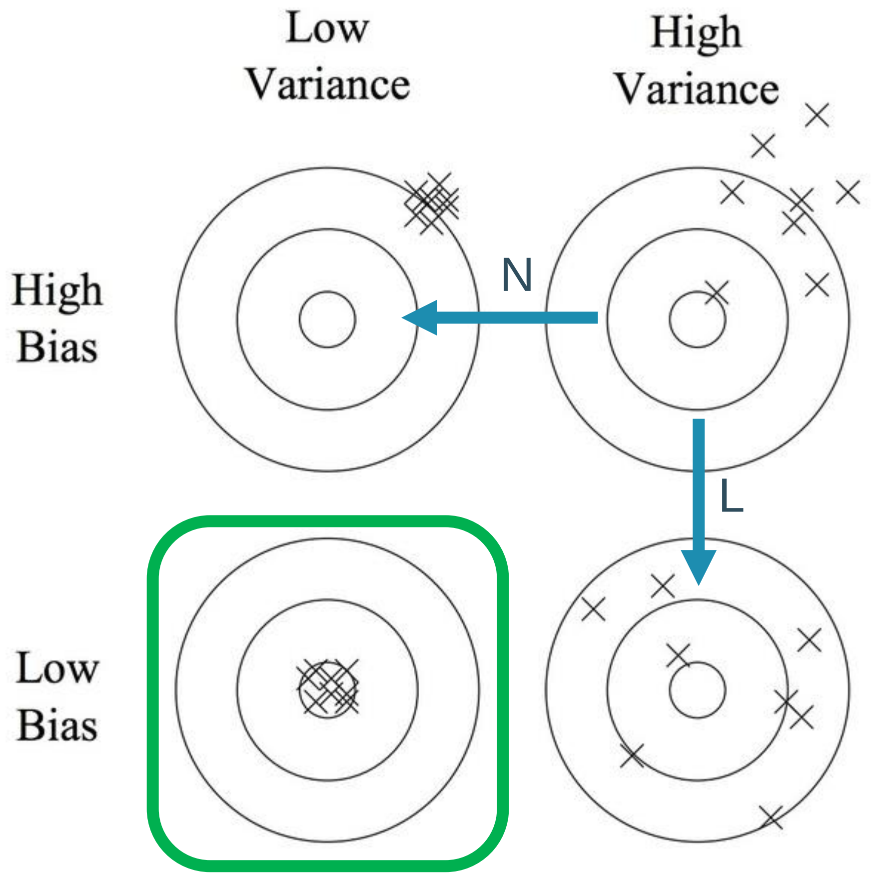
$$MSE = \sum_{l=0}^{L} Var[\Delta y_l] + (\mathbb{E}[y_L] - \mathbb{E}[y])^2$$

- Tolerance predefined: ε
- Goal: $MSE \leq \varepsilon^2$
- How many levels?

$$(\mathbb{E}[y_L] - \mathbb{E}[y])^2 < \frac{1}{2}\varepsilon^2$$ Bias constraint

- How many samples?

$$\sum_{l=0}^{L} Var[\Delta y_l] < \frac{1}{2}\varepsilon^2$$ Variance constraint



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Multilevel Quasi-Monte Carlo for interval analysis

- Multilevel Quasi-Monte Carlo applies to the two Cauchy parameters

$$\mathbb{E}(\delta)_L = \sum_{l=0}^{L} \mathbb{E}(\Delta\delta_l) \qquad \mathbb{E}(\gamma)_L = \sum_{l=0}^{L} \mathbb{E}(\Delta\gamma_l)$$

- Mean square error

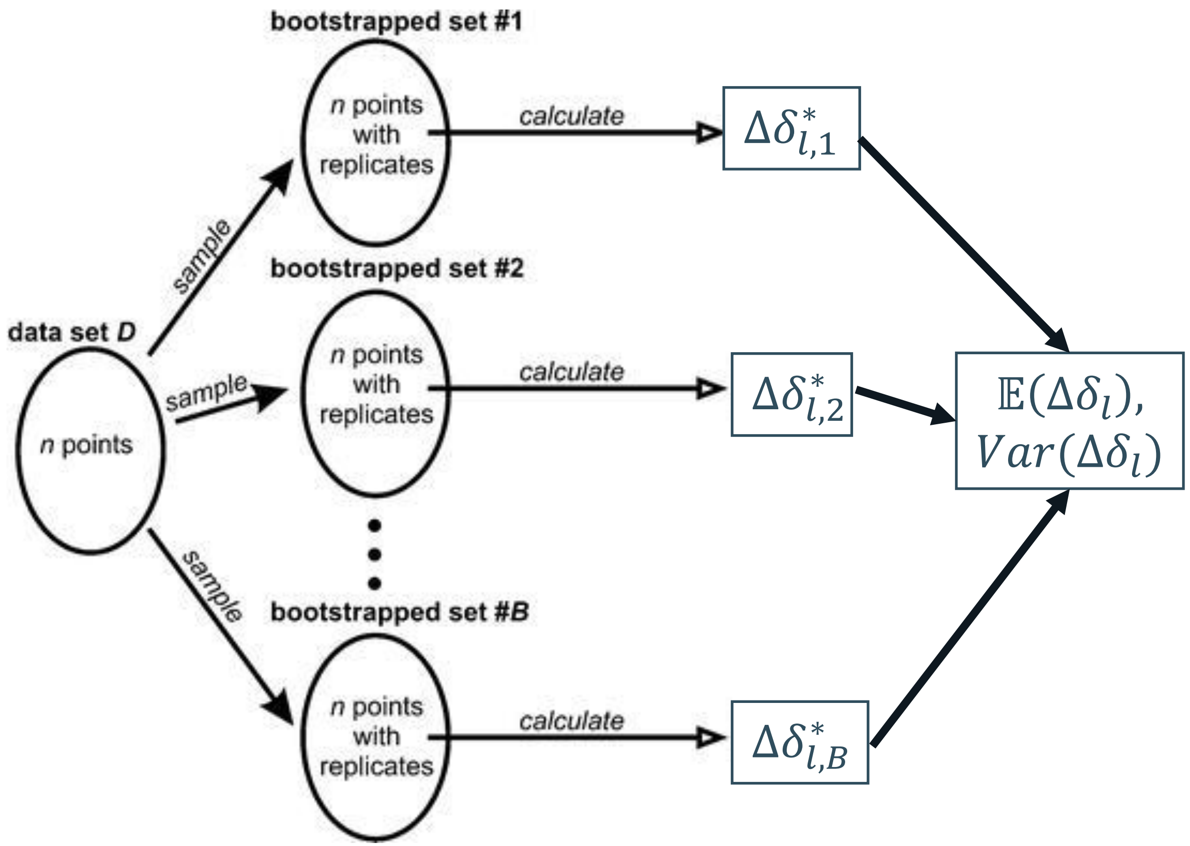
$$MSE = \sum_{l=0}^{L} Var[\Delta\delta_l] + (\mathbb{E}[\delta_L] - \mathbb{E}[\delta])^2 \qquad MSE = \sum_{l=0}^{L} Var[\Delta\gamma_l] + (\mathbb{E}[\gamma_L] - \mathbb{E}[\gamma])^2$$

- Variance
- Bias

Multilevel Quasi-Monte Carlo for interval analysis

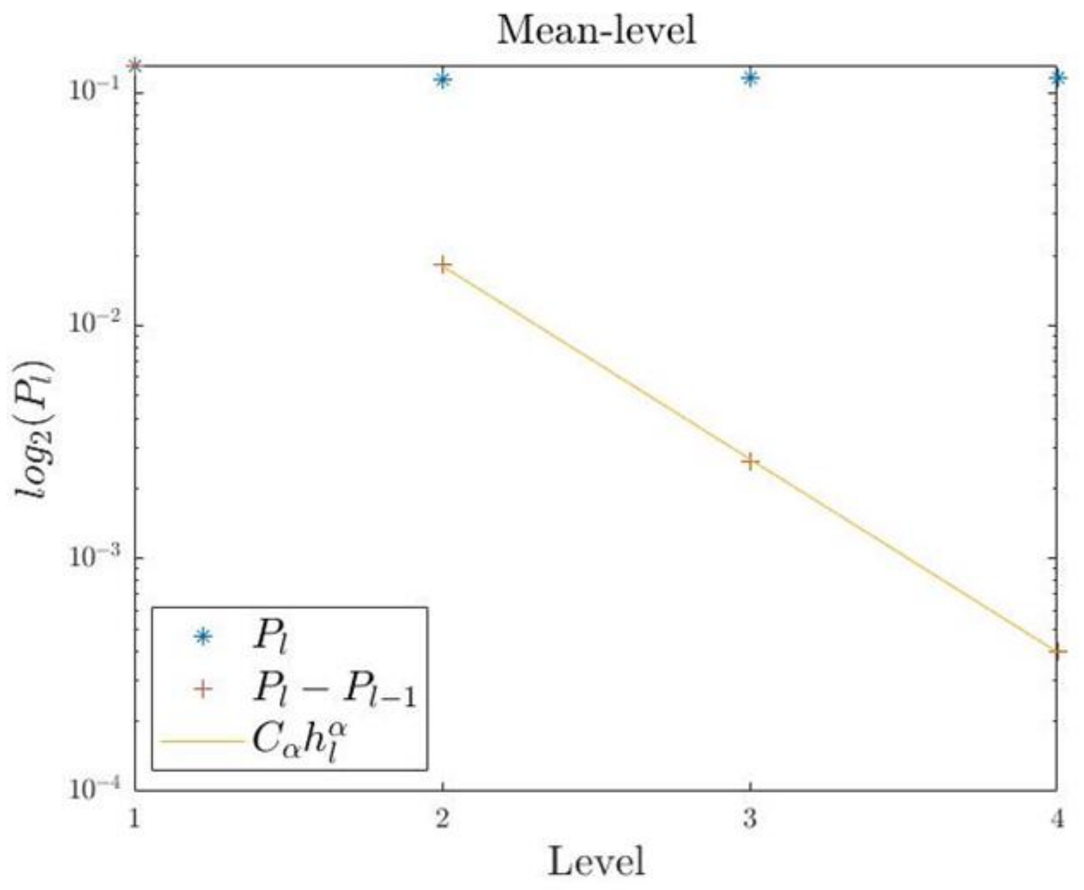
- Estimating variance of $\Delta\delta_l$ and $\Delta\gamma_l$
  - Median and Hodges-Lehmann Estimator
    - There is no direct variance formulation
  - Bootstrapping to estimate true variance
    - With B bootstrapping sets
    - Random set with replacement

- Variance ✓
- Bias ?



Multilevel Quasi-Monte Carlo for interval analysis

- Estimating bias of $\mathbb{E}[\delta_L]$and $\mathbb{E}[\gamma_L]$
  - Same procedure as Multilevel Monte Carlo
  - Extrapolation of bias to higher levels



- Variance ✓
- Bias ✓

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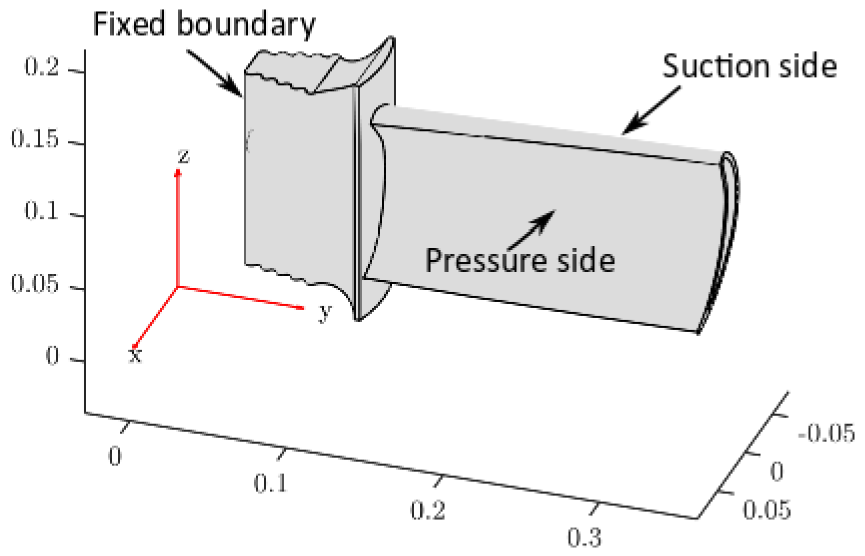
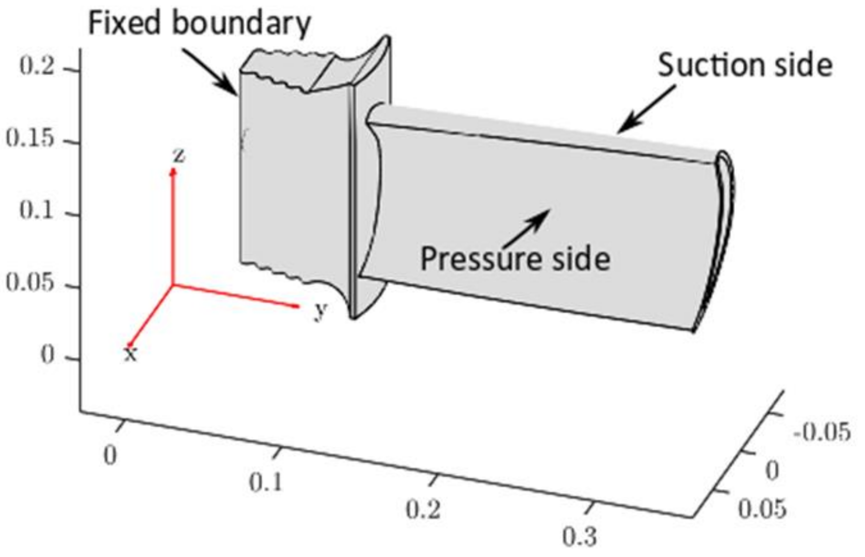
Numerical example

- Linear thermal-stress analysis of turbine blade Mathworks,Matlab 2019.
- Fixed Dirichlet boundary conditions on the bottom plane
- Mechanical loads
  - Pressure side
  - Suction side
- Uncertain parameters:
  - Pressure
  - Material parameters
  - Temperatures, convection coefficients



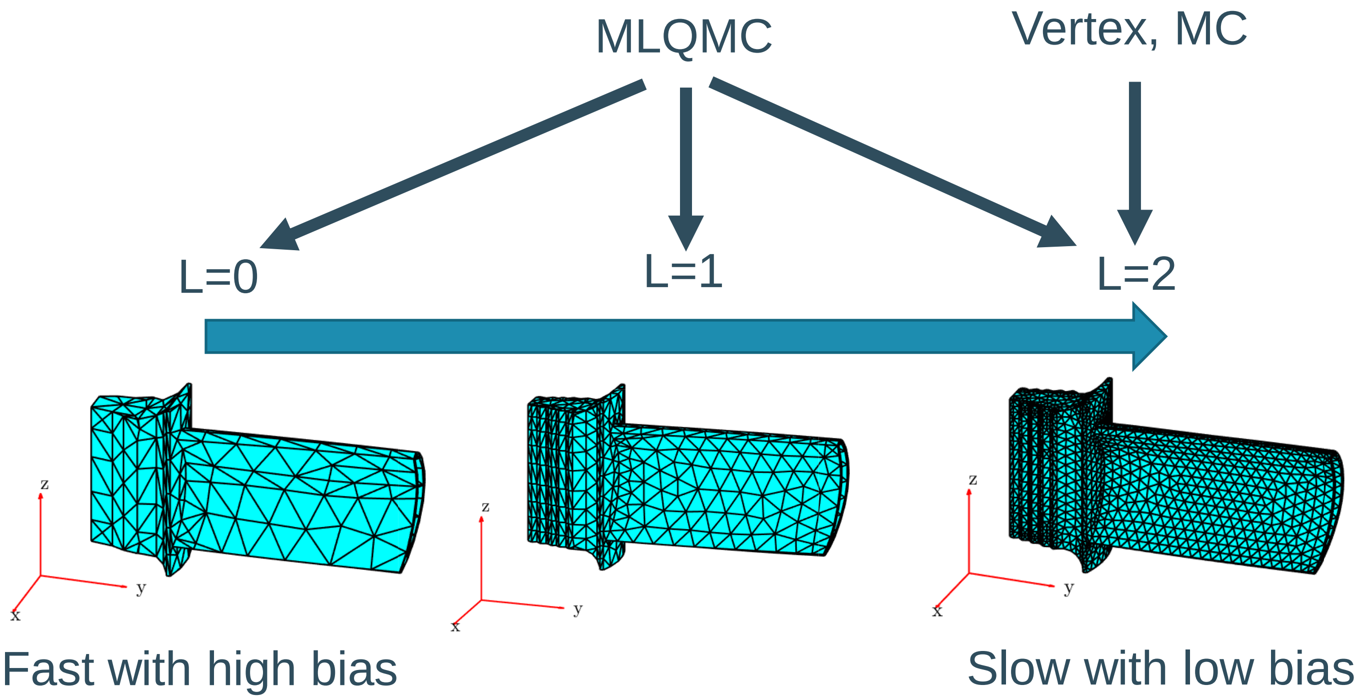
Numerical example

- Comparison of Interval analysis for increasing number of input intervals
  - Multilevel Quasi-Monte Carlo
  - Monte Carlo
  - Vertex Analysis
- Output:
  - Maximal displacement (-x) of the tip
  - For 8 and 20 input intervals



Numerical example:

- Multilevel Quasi-Monte Carlo:
  - Adaptive increasing levels
  - Initial sample size $N_i = 3$
  - L=0: 0,03m -> L=2: 0,0075m
- Vertex:
  - Accurate model
  - L=2: 0,075m
- Monte Carlo:
  - Accurate model
  - L=2: 0,075m



Numerical example: results

- Vertex:
  - Time: scales combinatorial: $2^8 = 256$ and $2^{20} = 1\,048\,576$
  - Accuracy: most accurate result (reference)
- Monte Carlo:
  - Time: lower than vertex
  - Accuracy: within relative tolerance 15%
- Multilevel Quasi-Monte Carlo:
  - Time: lower than vertex
  - Accuracy: within relative tolerance 15%

	8 interval scalars	
Vertex	$[0{,}33; 1{,}08]$mm	21min 20sec
MC	$[0{,}39; 1{,}01]$mm	17min 45sec
MLQMC	$[0{,}30; 1{,}04]$mm	16min 15sec
	20 interval scalars	
Vertex	–	65days
MC	$[0{,}14; 1{,}26]$mm	17min45sec
MLQMC	$[0{,}12; 1{,}32]$mm	17min20sec

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Conclusions

- **Multilevel Quasi-Monte Carlo** for interval analysis
- Computational **more efficient**: for medium to high number of input intervals
- **Accurate results** that are within the pre-defined tolerance
- Future work: non-linear models
- Acknowledgement
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