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An Evolutive Probability Transformation Method for the Dynamic Stochastic Analysis of Structures

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Background and Motivation

- The uncertainties are intrinsic in any engineering problem: mechanical/geometrical properties; **boundary conditions**, modeling errors; **loads**; imperfect road profiles; turbulence flows.
- Most physical behaviors exhibit appreciable randomness which can not be adequately represented by deterministic models.
- Stochastic differential equations are often used to model the stochastic dynamics of uncertain systems.
- The problem of the stochastic response determination of dynamic systems, once that the random properties of the uncertainties are assigned, is studied.
- An evolutive Probability Transformation Method (**EPTM**) for the study of the evaluation of the response of dynamic systems governed by first-order differential equations, characterized by uncertainties in the external actions and in their initial conditions will be presented.

Probability Transformation Method (PTM)

- The PTM is useful method for solving a large variety of random mechanic problems.

- PTM is a classical approach in stochastic mechanics (Papoulis, 1965), but it was rarely used in the past.

- It can be used to find the response characterization of stochastic systems directly through the knowledge of the corresponding probability density function (PDF).

- Most of the literature approaches deal with the evaluation of the moments and/or cumulants of the required response. But for non-Gaussian response a very high number of these statistical quantities are necessary, the convergence is not guaranteed and a strong computational effort is related to them.

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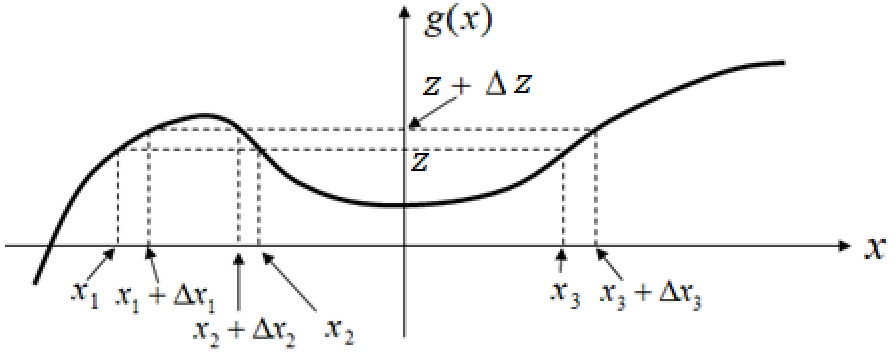
- The **PTM** is based on the theory of the space transformation laws of random vectors as well as on the principle of probability conservation. It gives a direct deterministic relationship between the joint PDFs of two random vectors related with each other by a deterministic law corresponding to the assigned space transformation.

- If $\mathbf{x}$ = random n-vector with JPDF $p_{\mathbf{x}}(\mathbf{x})$ and $\mathbf{g}(\cdot)$ = invertible n-function, such that $\mathbf{g}^{-1}(\cdot) \equiv \mathbf{h}(\cdot)$:

$$\mathbf{z} = \mathbf{g}(\mathbf{x}); \qquad \mathbf{x} = \mathbf{h}(\mathbf{z})$$



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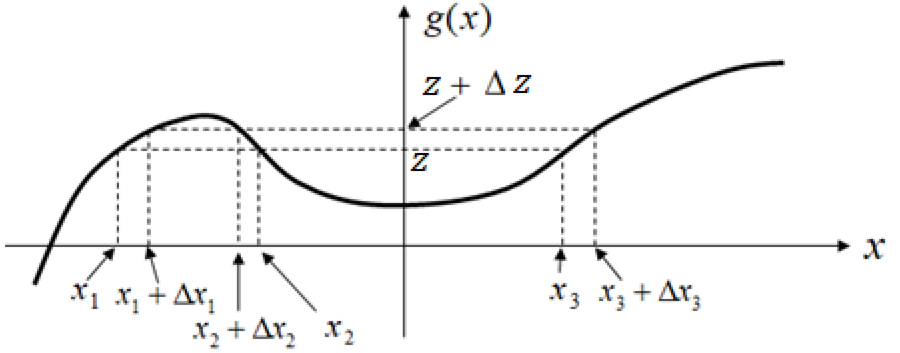
Then

$$p_{\mathbf{z}}(\mathbf{z}) = \frac{1}{\left|Det[\mathbf{J}_{\mathbf{g}}(\mathbf{z})]\right|} p_{\mathbf{x}}(\mathbf{h}(\mathbf{z}))$$

$\mathbf{J}_{\mathbf{g}}(\mathbf{z})$ = Jacobians of transformation



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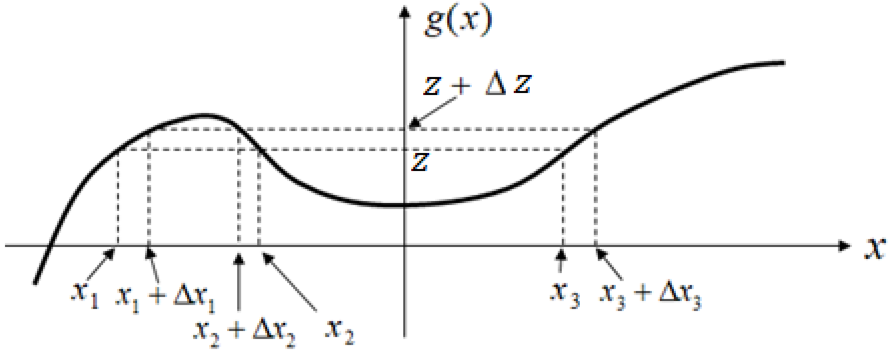

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$$p_{z_j}(z_j) = \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} p_{\mathbf{x}}(\mathbf{y}) \delta\left(z_j - h_j(\mathbf{y})\right) \mathrm{d}\, y_1 \cdots \mathrm{d}\, y_n$$

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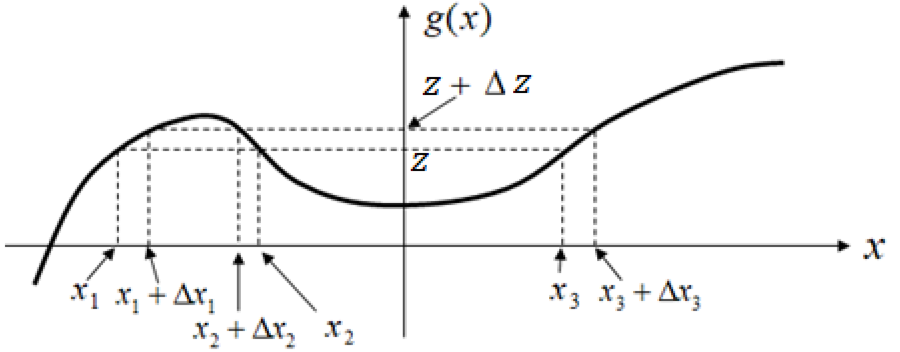

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- If $h_j(\mathbf{x}) = \boldsymbol{h}_j^T \boldsymbol{x}$ with $\boldsymbol{h}_j$ = *n*-vector of coefficients, PTM is given in terms of characteristic function (CF)

$$M_{z_j}(\omega_j) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} p_{\mathbf{x}}(\mathbf{y}) \exp(-\mathrm{i}\,\omega_j h_j(\mathbf{y})) \mathrm{d}\mathbf{y} = (2\pi)^{n-1} M_{x}(\boldsymbol{\theta})_{\boldsymbol{\theta}=\omega \boldsymbol{h}_j}$$

Stochastic first-order differential equation systems

1.1. First-order differential systems excited by a given random process

$$\dot{\mathbf{X}}(t) = \mathbf{A}\mathbf{X}(t) + \mathbf{B}\mathbf{F}(t), \quad \mathbf{X}(t_0) = \mathbf{X}_0$$

$\mathbf{X}(t)$ = is the n- random vector of the response state variables

$\mathbf{A}$ = is the $n \times n$ deterministic matrix which takes into account of the physical-geometrical characteristics of the system

$\mathbf{B}$ = is the $n \times m$ matrix defining the distribution of the external loads on the system

$\mathbf{F}(t)$ = is the m-random vector whose probabilistic characterization is known

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It is useful to write:

$$\mathbf{X}(t) = \mathbf{Y}_0(t) + \mathbf{Y}(t)$$

with $\mathbf{Y}_0(t) = \mathbf{\Theta}(t-t_0)\mathbf{X}_0, \quad \mathbf{Y}(t) = \int_{t_0}^{t} \mathbf{\Theta}(t-\tau)\mathbf{B}\mathbf{F}(\tau)\,\mathrm{d}\tau$

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- When $\mathbf{F}(t)$ is not a Gaussian vector process, the following vector process $\mathbf{Z}(\tau)$ is introduced:

$$\mathbf{Z}(\tau) = \mathbf{\Theta}(t-\tau)\mathbf{B}\mathbf{F}(\tau)$$

Its JPDF $p_{\mathbf{Z}}(\mathbf{z},\tau)$ can be easily obtained through the use of the PTM rules to the linear transformation $\mathbf{F}(\tau) \rightarrow \mathbf{Z}(\tau)$

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Hence

$$\mathbf{Y}(t) = \int_{t_0}^{t} \mathbf{Z}(\tau)\,\mathrm{d}\tau$$

At this point, the last step to do is the probabilistic characterization of the random vector process $\mathbf{Y}(t)$ once that the characterization of the vector process $\boldsymbol{Z}(\tau)$ has been found.

Stochastic first-order differential equation systems

1.1. First-order differential systems excited by a given random process

- Do to the time integral relationship between these two processes, an efficient numerical solution can be obtained by using the following theorem on the JCFs of $\boldsymbol{Y}(t)$ and $\boldsymbol{Z}(t)$:

THEOREM 1. *((Soong, 1973) If the m.s. integral* $\mathbf{X}(t)$, $t \in T$, *exists, then:*

$$M_{\mathbf{Y}}(\omega, t) = \lim_{m\to\infty \Delta m\to 0} M_{\mathbf{Z}}(\omega\left(\tau_1 - \tau_0\right), \tau_1'; \cdots; \omega\left(\tau_m - \tau_{m-1}\right), \tau_m') \quad \tau_j' \in \left(\tau_{j-1}, \tau_j\right)$$

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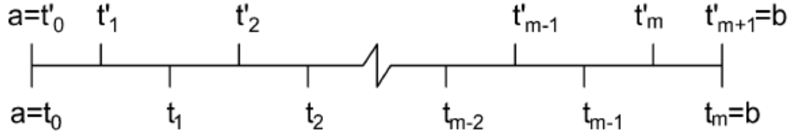
- Employing some of the theory of random differential equations and assuming that $\boldsymbol{Y}(t)$ is a mean square Riemann integral over a generic interval [a; b], it can be written:



$$\int_a^b \mathbf{Y}(t)dt \equiv \lim_{\Delta m \to 0} \sum_{j=1}^{m} \mathbf{Y}(t_j')\Delta t_j$$

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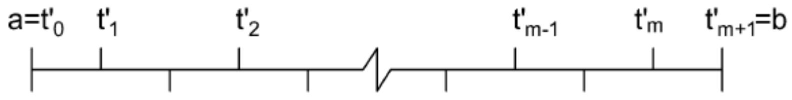

Figure 1.: Finite partitions of an interval $[a, b]$.

- By truncating the value of m to a sufficiently high value, the expression of this theorem becomes an efficient way to evaluate numerically the response CF. Then, the response PDF is obtained by the Fourier anti-transform of the response CF.

- The application of the PTM with the fundamental results of the above Theorem gives a stochastic procedure, here called **EPTM**, able to characterize the response of dynamical systems in terms of evolutive PDF.

Stochastic first-order differential equation systems

1.2. First-order differential systems with random load and random initial conditions

- Now, besides the loads acting on the system, even the initial conditions are assumed to be random variables. The **EPTM** approach in the assumption of independence between these two kinds of actions will be shown.

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- The expression of the response vector can be rewritten as follows:

$$\mathbf{X}(t) = \bar{\mathbf{X}}_0(t) + \bar{\mathbf{X}}_F(t)$$

$\mathbf{X}_0$ and $\bar{\mathbf{X}}_F(t)$ being two independent random processes given by:

$$\begin{aligned} \bar{\mathbf{X}}_0(t) &= \mathbf{\Theta}(t)\mathbf{X}_0; \\ \bar{\mathbf{X}}_F(t) &= \int_0^t \mathbf{\Theta}(t-\tau)\mathbf{B}\mathbf{F}(\tau)d\tau \end{aligned}$$

- Each of these processes can be easily characterized through the evaluation of their CFs.

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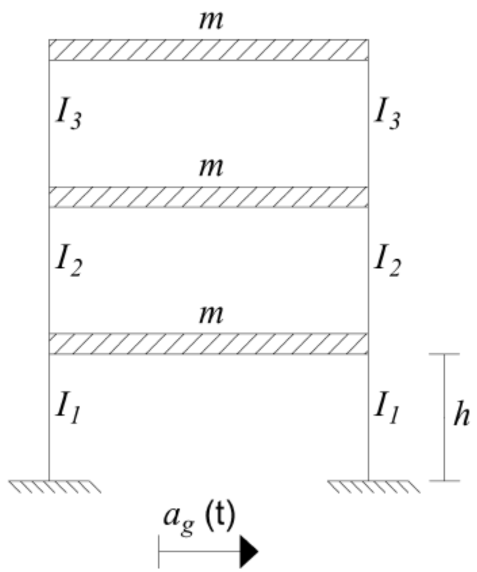
- Each of these processes can be easily characterized through the evaluation of their CFs
- The response CF $M_{\mathbf{x}}(t)$ can be obtained by considering an important property of the CF of the sum of two independent random processes

$$M_{\mathbf{X}(t)}(\omega, t) = M_{\bar{\mathbf{X}}_0(t)}(\omega, t) M_{\bar{\mathbf{X}}_F(t)}(\omega, t)$$

Then, the response PDF is obtained by the Fourier anti-transform of the response CF.

Numerical example



Shear-type plane system

$$\begin{aligned}\mathbf{M}\ddot{\mathbf{U}}(t)+\mathbf{C}\dot{\mathbf{U}}(t)+\mathbf{K}\mathbf{U}(t) &= \mathbf{F}(t)\\ \mathbf{U}(0) &= \mathbf{U}_0\end{aligned}$$

By introducting the state vector $\mathbf{X}^T(t)=(\mathbf{U}^T(t)\ \ \dot{\mathbf{U}}^T(t))$, the system's equations can be converted into the following first-order differential system:

$$\begin{aligned}\dot{\mathbf{X}}(t) &= \mathbf{D}\mathbf{X}(t)+\mathbf{v}\mathbf{F}(t);\\ \mathbf{X}(0) &= \mathbf{X}_0\end{aligned}$$

The response $\mathbf{X}(t)$ is evaluated takinh into account the Duhamel integral, that is:

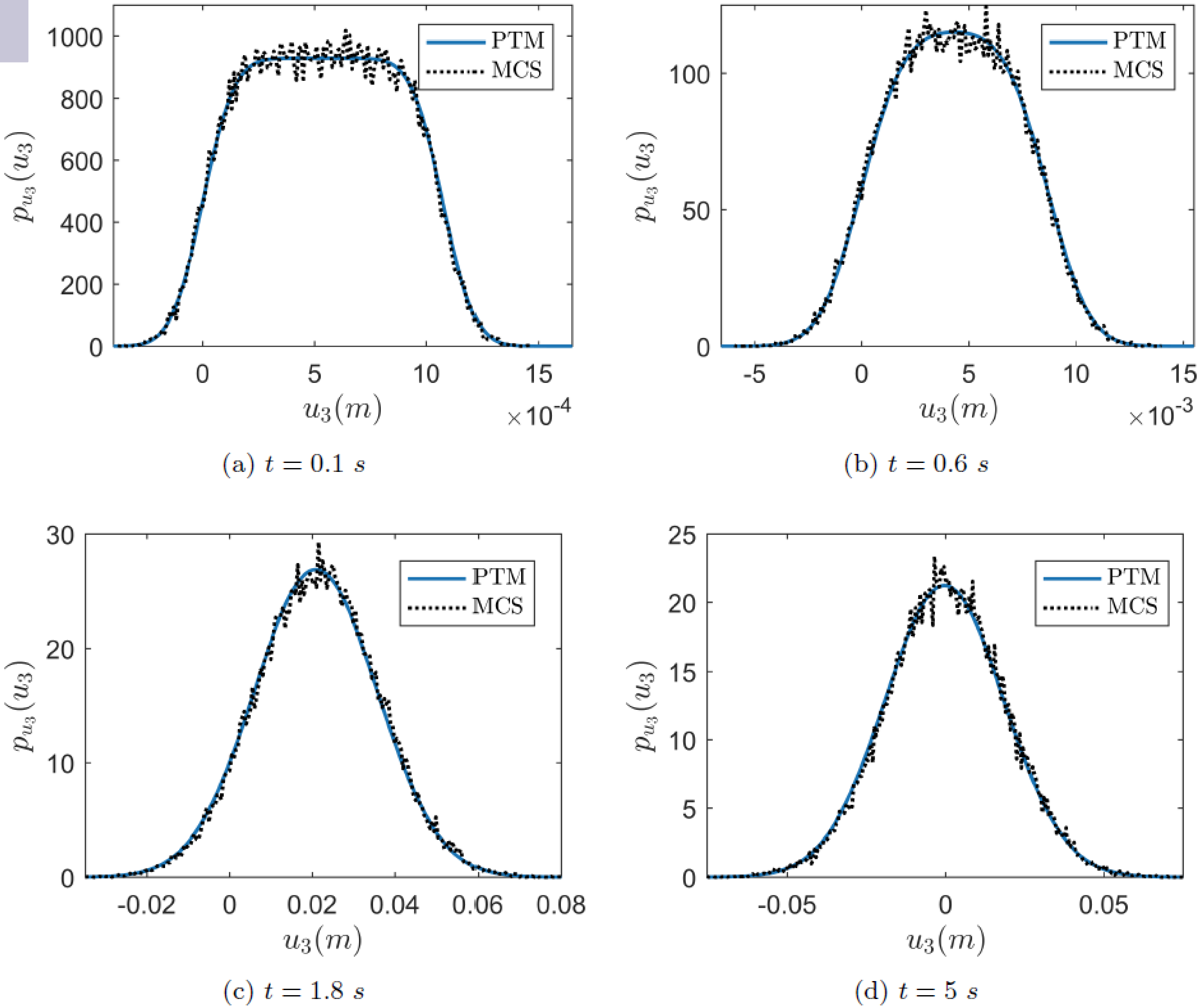
$$\mathbf{X}(t)=\mathbf{\Theta}(t)\mathbf{X}_0+\int_0^t \mathbf{\Theta}(t-\tau)\mathbf{v}\mathbf{F}(\tau)d\tau$$

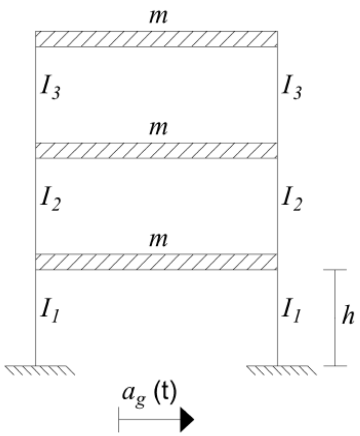
The system is forced by a zero-mean Gaussian stationary ground acceleration, defined by its one side Clough-Penzien spectra density

$$S_a(\omega)=\frac{\omega_r^4+4\xi_r^2\omega_r^2\omega^2}{(\omega_r^2-\omega^2)^2+4\xi_r^2\omega_r^2\omega^2}\,\frac{\omega^4}{(\omega_p^2-\omega^2)^2+4\xi_p^2\omega_p^2\omega^2}\,\frac{0.141\xi_r^2a_{g0}^2}{\omega_r\sqrt{1+r\xi_r^2}}$$

While $\mathbf{X}_0$, is assumed as a random vector described by random variables uniformly distributed with $\sigma_{x0} = 0.15$

Numerical example





Displacement PDF evaluated for four different instants. PTM (continuous line); MCS (dashed line).

Conclusions

- A stochastic procedure for the dynamic analyses of systems characterized by uncertainties in the external actions and in their initial conditions has been presented.

- Based on PTM, the EPTM combines the main properties of the mean square random calculus with the principle of conservation of probability.

- The proposed procedure allows to preserve the system's probability in time step by step and gives the random response directly in terms of CFs; then the response PDF of the system can be evaluated easily by the inverse Fourier transform.

- The application of the EPTM for the stochastic dynamic analysis of a shear-type system has been confirmed the goodness of the results in terms of evolutive PDF.

Thanks!