Why Ellipsoids in Mechanical Analysis of Wood Structures

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1. Formulation of the Problem

- Many constructions are made of wood.
- Wood is one of the oldest materials used in construction.
- During the past millennia, people have developed a lot of skills for working with wood.
- However, in spite of this experience, wood remains one of the most difficult materials to handle.
- The main reason for this difficulty is that:
  - in contrast to many other construction materials which are mostly homogeneous and isotropic,
  - wood is highly inhomogeneous and anisotropic.

2. Formulation of the Problem (cont-d)

- At each point in the wooden beam:
  - both the average values and fluctuations of the local mechanical properties
  - depend on whether the direction is longitudinal, radial or tangential with respect to the grain.
- In designing wooden constructions, it is important:
  - to properly describe and to properly take into account
  - this inhomogeneity and anisotropy.
- How can we describe local fluctuations of mechanical characteristics?
- These fluctuations are caused by many different relatively small factors.

3. Formulation of the Problem (cont-d)

- It is known that the distribution of the joint effect of a large number of small factors is close to Gaussian.
- This follows from the Central Limit Theorem, according to which:
  - this distribution tends to Gaussian
  - when the number of factors increases.
- To describe a Gaussian distribution, it is sufficient to describe its first and second moments.
- For a general random field $f(x)$, this means that we need to describe:
  - its mean values $E[f(x)]$ (where $E[\cdot]$ denotes the expected value) and
  - its covariances $E[f(x) \cdot f(y)]$.

4. Formulation of the Problem (cont-d)

- For fluctuations, the mean is 0, so we only need to describe covariances.
- In statistics, it is often convenient:
  - instead of explicitly describing covariances,
  - to describe the standard deviations and correlations:

$$\sigma[f(x)] \stackrel{\text{def}}{=} \sqrt{E[(f(x)^2]};\ \ \rho(x,y) \stackrel{\text{def}}{=} \frac{E[f(x)\cdot f(y)]}{\sigma[f(x)]\cdot\sigma[f(y)]}.$$

- Then, covariances can be reconstructed as

$$E[f(x)\cdot f(y)] = \sigma[f(x)]\cdot\sigma[f(y)]\cdot\rho(x,y).$$

- An interesting property of the corresponding correlation functions was recently empirically found.

5. Formulation of the Problem (cont-d)

- This property is about:
  - *iso-correlation* surfaces corresponding to each spatial location x,
  - i.e., surfaces formed by all the points y for which the correlation $\rho(x, y)$ is equal to a constant ρ_0.
- Empirical analysis shows that:
  - for each point x,
  - the corresponding surfaces are well approximated by concentric homothetic ellipsoids.
- This property helps narrow down possible functions $\rho(x, y)$ when we analyze mechanical properties of wood.
- Thus, it has a potential to make mechanical analysis of wooden structures more efficient.

6. Formulation of the Problem (cont-d)

- The problem is that so far, this property was purely empirical, it had no theoretical justification.
- Thus, engineers were reluctant to use it.
- It is known that sometimes:
  - empirical properties found under some conditions
  - do not work well when conditions change.
- We want to make this property more reliable and thus, more practically useful.
- It is therefore desirable to come up with a theoretical explanation.
- In this talk, we provide a desired theoretical explanation for this empirical fact.

7. Our Explanation: Main Idea

- We show that there exists the smallest dimension d for which:
  - it is possible to have an affine-invariant optimality criterion
  - on the space of all such d-dimensional classes.
- We also show that for any such criterion, the optimal family consists of concentric homothetic ellipsoids.
- Thus, such families of ellipsoids provide the optimal approximation to the actual surfaces:
  - at least in the *first* approximation, i.e.,
  - approximation corresponding to the smallest possible number of parameters.

8. Family of Sets: Towards a Precise Definition

- For each spatial point x, we would like to describe:
  - for each possible value ρ_0 of the correlation $\rho(x, y)$,
  - the set $S_{\rho_0}(x) = \{y : \rho(x, y) \geq \rho_0\}$.
- What are the natural properties of these families of sets?
- The first property is coverage.
- For each y, there is some value of $\rho(x, y)$.
- So for this x, the union of all these sets $S_{\rho_0}(x)$ coincides with the whole space.
- The second property is monotonicity.
- If $\rho(x, y) \geq \rho_0$ and $\rho_0 \geq \rho'_0$, then $\rho(x, y) \geq \rho'_0$.
- So, the sets $S_{\rho_0}(x)$ should be inclusion-monotonic:

$$\text{if } \rho_0 \leq \rho'_0, \text{ then } S_{\rho'_0}(x) \subseteq S_{\rho_0}(x).$$

9. Family of Sets (cont-d)

- The third property is boundedness.
- From the physical viewpoint:
  - the further away is the point y from the point x,
  - the less the physical quantities corresponding to these points are correlated.
- As the distance increases, this correlation should tend to 0.
- Thus, each set $S_{\rho_0}(x)$ is *bounded*.
- The fourth property is continuity.
- In physics:
  - most processes are continuous,
  - with the exception of processes like fracturing, which we do not consider here.

10. Family of Sets (cont-d)

- We can therefore conclude that the correlation $\rho(x, y)$ continuously depends on y, so:
  - if we have $\rho(x, y_n) \geq \rho_0$ for some sequence of points y_n that converges to a point y $(y_n \to y)$,
  - then we should have $\rho(x, y) = \lim\limits_{n\to\infty} \rho(x, y_n) \geq \rho_0$.
- Thus, if $y_n \in S_{\rho_0}(x)$ and $y_n \to y$, then $y \in S_{\rho_0}(x)$.
- So, each set $S_{\rho_0}(x)$ is *closed*.
- Similarly, it is reasonable to conclude that the set $S_{\rho_0}(x)$ should continually depend on ρ_0:
  - if the two values ρ_0 and ρ'_0 are close,
  - then the corresponding sets $S_{\rho_0}(x)$ and $S_{\rho'_0}(x)$ should also be close.
- A natural way to describe closeness between (bounded closed) sets is to use the so-called Hausdorff distance.

11. Family of Sets (cont-d)

- We say that the sets A and B are ε-*close* if:
  - every point $a \in A$ is ε-close to some point $b \in B$, i.e., $d(a, b) \leq \varepsilon$, and
  - every point $b \in B$ is ε-close to some point $a \in A$.
- The Hausdorff distance $d_H(A, B)$ is defined as the smallest ε for which the sets A and B are ε-closed.
- It can be shown that this distance can be equivalently defined as follows:

$$d_H(A, B) = \max\left(\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\right), \text{ where}$$

$$d(a, B) \stackrel{\text{def}}{=} \inf_{b \in B} d(a, b).$$

- What is the set of possible values of the parameter?
- In this family of sets, correlation value is a parameter.

12. Family of Sets (cont-d)

- Correlations can take any value from -1 to 1.
- When $y = x$, the correlation is clearly equal to 1.
- When $y \to \infty$, we get values close to 0.
- Since the function $\rho(x, y)$ is continuous, this function takes all intermediate values.
- So, the possible values of the correlation form some interval.
- In some cases, we may have all possible negative values.
- In other cases, only some negative values, in yet other cases, we only have non-negative values.
- So, in general, we will consider all possible intervals of possible value of ρ_0.
- This interval may be closed – if there are points with limit correlation, or is can be open.

13. Definition

- So, we arrive at the following definition.
- Let $N \geq 2$ be an integer.
- Let I be an interval.
- By a *family of sets*, we mean a set $\{S_c : c \in I\}$ of bounded closed sets $S_c \subseteq \mathbb{R}^N$ for which:
  - the dependence of S_c on c is continuous: if $c_n \to c$, then $d_H(S_{c_n}, S_c) \to 0$;
  - the family S_c is monotonic: if $c < c'$, then $S_{c'} \subseteq S_c$; and
  - the union of all the sets S_c coincides with the whole space.

14. Comments

- According to this definition, the family remains the same if we simply re-parameterize the family.
- For example:
  - instead of the original parameter c,
  - we can use a new parameter $c' = c + c_0$ or $c' = \lambda \cdot c$ for some constants c_0 and λ.
- In our specific problem, we are interested in the 3-D case $N = 3$.
- However, we can envision similar problem in the plane $N = 2$ or in higher-dimensional spaces.
- So, in this talk, we consider the general case $N \geq 2$.

15. Comments (cont-d)

- We are specifically interested:
  - in *concentric homothetic families of ellipsoids*, i.e.,
  - in families of the type $S_c = c \cdot E + a$, where a is a given vector, and E is an ellipsoid with center 0.

16. Class of Families of Sets

- For different situation, in general:
  - we get different correlations and thus,
  - we get different families of sets.
- We would like to find a general class of such families that would, ideally, cover all such situations.
- We can use different parameters to differentiate different families from this class.
- In other words, a class can be described as a method for assigning:
  - to each possible combination of values of these parameters,
  - a specific family.
- As before, it makes sense to require that the resulting mapping is continuous.

17. Class of Families of Sets (cont-d)

- Here is a precise definition.
- Let $N \geq 2$ and $r > 0$ be integers.
- By an *r-parametric class of families of sets*, we mean a mapping that assigns,
  - to each element $p = (p_1, \ldots, p_r)$ from an open r-dimensional set $D \subseteq \mathbb{R}^r$,
  - a family $\{S_c(p)\}$ so that the dependence of $S_c(p)$ on c and p is continuous.

18. Optimality Criteria: General Idea

- Out of all possible classes, we want to select a class which is, in some reasonable sense, optimal.
- For this, we need to be able to describe when some classes are better than others.
- In other words, we need to have an *order* on the set of all the classes.
- It would be nice to have a *total* (*linear*) order, in the sense that:
  - for every two classes,
  - we should be able to tell which one is better.
- However, it may be sufficient to have a *partial* order – as long as this order enables us to select the best class.
- It is OK if for some not-best classes, we do not have an opinion of which of them is better.

19. Optimality Criteria: General Idea (cont-d)

- In practice, usually, optimality criteria are described in numerical form:
  - we have an objective function $f(a)$ that assigns a numerical value to each possible alternative a, and
  - we want to select an alternative for which this value is the largest possible,
  - or, depending on the context, the smallest possible.
- For example:
  - a company wants to maximize its profit,
  - a city wants to upgrade its road system so as to minimize the average travel time, etc.
- However, often, we need to go somewhat beyond this approach.

20. Optimality Criteria: General Idea (cont-d)

- For example, a company may have two (or more) different projects that lead to the same expected profit.
- In this case, we can use this non-uniqueness to optimize something else.
- For example:
  - out of all most-profitable projects,
  - we can select the one that leads to the smallest possible long-term environmental impact.
- In this case, we have a more complex criterion for comparing alternatives: we say that a is better if:
  - either $f(a) > f(a')$
  - or $f(a) = f(a')$ and $g(a) > g(a')$, for some other numerical criterion $g(a)$.

21. Optimality Criteria: General Idea (cont-d)

- If this still does not select us a unique alternative, we can optimize yet something else, etc.
- In view of this possibility, in this talk, we do not restrict ourselves to numerical optimization criteria.
- Instead, we use the most general definition of the optimality criterion, when:
  - for some pairs of alternatives a and a', we know that a is better (we will denote it by $a' < a$),
  - for some pairs of alternatives a and a', we know that a' is better ($a < a'$), and
  - for some pairs of alternatives a and a', a and a' are of the same value (we will denote it by $a \sim a'$).
- Clearly, if a' is better than a, and a'' is better than a', then a'' should be better than a, etc.

22. Optimality Criteria: General Idea (cont-d)

- Thus, we arrive at the following definition
- Let A be a set; elements of this set will be called *alternatives*.
- By an *optimality criterion*, we mean a pair of binary relations $(<, \sim)$ on the set A for which:
  - if $a < a'$ and $a' < a''$, then $a < a''$;
  - if $a < a'$ and $a' \sim a''$, then $a < a''$;
  - if $a \sim a'$ and $a' < a''$, then $a < a''$;
  - if $a \sim a'$ and $a' \sim a''$, then $a \sim a''$;
  - if $a \sim a'$, then $a' \sim a$;
  - if $a < a'$, then we cannot have $a' < a$ or $a \sim a'$.
- Such a pair of relations is sometimes called a *partial pre-order*.

23. Optimality Criteria: General Idea (cont-d)

- Let $(<, \sim)$ be an optimality criterion on a set A.
- An alternative a_{opt} is called *optimal* with respect to this criterion if for every alternative $a \in A$, we have

$$a < a_{\text{opt}} \text{ or } a \sim a_{\text{opt}}.$$

24. We Need A Final Optimality Criterion

- For the optimality criterion to be useful, it must select *at least one* optimal alternative.
- If the criterion selects *several* alternatives as optimal, this means that this criterion is not final.
- We can use the resulting non-uniqueness:
  - to optimize something else,
  - i.e., in effect, to come up with a better optimality criterion.
- If for this better criterion, we still have several optimal alternatives, we should modify this criterion again.
- Finally, we get a criterion for which there is exactly one optimal alternative.
- We will call such criteria *final*.

25. For Our Problem, an Optimality Criterion Must Be Affine-Invariant

- In our case, we want to compare different classes (of families of sets).
- In selecting optimality criteria, it is reasonable to take into account that:
  - while we want to deal with sets of points in physical space,
  - from the mathematical viewpoint, we deal with sets of tuples of real numbers.
- Real numbers (coordinates) describing each point depend on what coordinate system we use.
- If we select a different starting point, then all the coordinates are shifted $x_i \to x_i + a_i$.

26. Affine-Invariant (cont-d)

- If we select different axes for the coordinates, we get a rotation $x_i \to \sum\limits_{j=1}^{N} r_{ij} \cdot x_j$ for an appropriate matrix r_{ij}.
- These transformations make sense for the *isotropic* case, when:
  - all the properties of a material
  - are the same in all directions.
- Wood is an example of an *anisotropic* material.
- For example, it is easier to cut it along the orientation of the original tree than across that orientation.
- It is known that in many cases:
  - the description of an anisotropic material can be reduced to the isotropic case
  - if we apply an appropriate affine transformation.

27. Affine-Invariant (cont-d)

- This usually comes from the fact that, e.g.:
  - mechanical properties of a body can be described by a symmetric matrix, and
  - a symmetric matrix becomes symmetric if we use its eigenvectors as new axes.
- In view of this, it is reasonable to require that our optimality criterion is invariant:
  - not only with respect to shifts and rotations,
  - but also with respect to all possible affine (linear) transformations.
- Thus, we arrive at the following definitions.
- Let $N > 2$ be an integer.

28. Affine-Invariant (cont-d)

- By an *affine transformation*, we mean $(Tx)_i = a_i + \sum\limits_{j=1}^{N} b_{ij} \cdot x_j$ for some reversible matrix b_{ij}.
- Let T be an affine transformation.
- Let $S \subseteq \mathbb{R}^N$ be a set.
- By the *result* $T(S)$ of applying T to S, we mean the set $\{T(s) : s \in S\}$.
- Let $F = \{S_c : c \in I\}$ be a family of sets.
- By the *result* $T(F)$ of applying T to F, we mean the family $\{T(S_c) : c \in I\}$.
- Let $C = \{S_c(p)\}$ be class of families.
- By the *result* $T(C)$ of applying T to C, we mean the class $\{T(S_c(p))\}$.

29. Affine-Invariant (cont-d)

- Let A be a set of alternatives, let $(<, \sim)$ be an optimality criterion of the set A.
- Let $\mathcal{T}$ be a class of transformations $A \to A$.
- We say that $(<, \sim)$ is $\mathcal{T}$*-invariant* if for all $T \in \mathcal{T}$ and $a, a' \in A$, we have:
  - if $a < a'$ then $T(a) < T(a')$, and
  - If $a \sim a'$, then $T(a) \sim T(a)$.

30. Main Result

- Let $N > 0$ and $r > 0$ be integers.
- We consider sets in $\mathbb{R}^N$.
- Let $(<, \sim)$ be a final affine-invariant optimality criterion on the set of all r-parametric classes of families.
- Then $r \geq r_{\min} \stackrel{\text{def}}{=} \frac{N \cdot (N+3)}{2} - 1$, and:
  - for $r = r_{\min}$,
  - the optimal class consists of concentric homothetic families of ellipsoids.
- This result indeed shows that:
  - the class of concentric homothetic families of ellipsoids
  - is the simplest (= fewer parameters) of all possible optimal classes.

31. Proof

- Since the optimality criterion is final, there exists exactly one optimal class C_{opt} for which:

$$C < C_{\text{opt}} \text{ or } C \sim C_{\text{opt}} \text{ for all classes } C.$$

- Let us prove that the optimal class C_{opt} is itself affine-invariant, i.e., that $T(C_{\text{opt}}) = C_{\text{opt}}$ for each affine T.
- Indeed, due to optimality, for each class C and for each affine transformation class T, for $T^{-1}(C)$, we have:

$$T^{-1}(C) < C_{\text{opt}} \text{ or } T^{-1}(C) \sim C_{\text{opt}}.$$

- Since the criterion is affine-invariant, we have:

$$T(T^{-1}(C)) < T(C_{\text{opt}}) \text{ or } T(T^{-1}(C)) \sim T(C_{\text{opt}}).$$

- Here, by the definition of the inverse transformation:

$$T(T^{-1}(C)) = C.$$

32. Proof (cont-d)

- So we conclude that for every class C, we have:

$$C < T(C_{\text{opt}}) \text{ or } C \sim T(C_{\text{opt}}).$$

- By definition of optimality, this means that the class $T(C_{\text{opt}})$ is optimal.
- However, our optimality criterion is final, which means that there is only one optimal class.
- Thus, indeed, $T(C_{\text{opt}}) = C_{\text{opt}}$.
- Since the optimal class is affine-invariant, with each family F this class also contains the family $T(F)$.
- This means that for each set S_c from each family, some family from the optimal class contains the set $T(S_c)$.
- Let us show that $r \geq \dfrac{N \cdot (N+3)}{2} - 1$.

33. Proof (cont-d)

- Indeed, it is known that:
  - for every non-degenerate bounded set S (i.e., not contained in a proper subspace),
  - among all ellipsoids that contain S, there exists a unique ellipsoid of the smallest volume.
- It is also known that this correspondence between a set and the corresponding ellipsoid is affine-invariant:
  - if an ellipsoid E corresponds to the set S_c, then,
  - for each affine transformation T, to the set $T(S_c)$ there corresponds the ellipsoid $T(E)$.
- It is known that every two ellipsoids can be obtained from each other by an affine transformation.

34. Proof (cont-d)

- Thus:
  - the family of all ellipsoids corresponding to all the sets from all the families
  - consists of all the ellipsoids.
- How many ellipsoids are there?
- A general ellipsoid can be determine by a quadratic formula $\sum\limits_{ij} a_{ij} \cdot x_i \cdot x_j + \sum\limits_{i=1}^{N} a_i \cdot x_i \le 1.$
- Here, a_{ij} is a symmetric matrix a_{ij} and a_i is a vector.
- It is easy to see that different combinations of the matrix and the vector lead to different ellipsoids.
- We need N values $a_1, \ldots, a_N$ to describe a vector.

35. Proof (cont-d)

- Out of N^2 elements of the matrix:
  - we need N values to describe its diagonal values a_{ii}, and
  - we need $\dfrac{N^2 - N}{2}$ to describe non-diagonal elements.
- We divide by two since the matrix is symmetric:

$$a_{ij} = a_{ji}.$$

- Thus, overall, we need

$$N + N + \frac{N^2 - N}{2} = \frac{N \cdot (N+3)}{2} \text{ values.}$$

- So, the set of all ellipsoids is:

$$\frac{N \cdot (N+3)}{2}\text{-dimensional.}$$

36. Proof (cont-d)

- To each set S_c from families from the optimal class, we assign an ellipsoid.
- Thus, the dimension of the set of such sets should also be at least $\dfrac{N\cdot(N+3)}{2}$-dimensional.
- These sets are divided into 1-parametric families.
- So the dimension r of the class of such families cannot be smaller than the above dimension minus 1.
- Thus, indeed, $r \geq \dfrac{N\cdot(N+3)}{2} - 1.$

37. Proof (cont-d)

- Let us now prove that:
  - for the smallest possible dimension
$$r = r_{\min} \stackrel{\text{def}}{=} \frac{N \cdot (N+3)}{2} - 1,$$
  - all the sets S_c from the each family of the optimal class are ellipsoids.
- Indeed, we showed that each ellipsoid is associated with some set S_c from one of these families.
- The unit ball with a center at 0 is clearly an ellipsoid.
- Let us consider the set S_c which is associated with this unit ball.
- A unit ball is invariant with respect to all the rotations around its center.

38. Proof (cont-d)

- If the associated set S_c is not equal to the unit ball, this means that:
  - this set is not invariant
  - with respect to at least some rotations.
- In other words:
  - the group of all rotations that leave this set invariant
  - is a proper subgroup of the group of all rotations.
- This implies that the dimension of this group is smaller than the dimension of the group of all rotations.
- Thus, that there exists at least 1-parametric family $\mathcal{R}$ of rotations R w.r.t. which the set S_c is not invariant.
- The optimal class is affine-invariant.

39. Proof (cont-d)

- Thus, all the sets $R(S_c)$ are also sets from some family from the optimal class.
- For all these sets, the same unit ball is the smallest-volume ellipsoid.
- Thus, for this particular ellipsoid – the unit ball:
  - we have at least a 1-dimensional family of sets S_c
  - associated with this ellipsoid.
- By applying a generic affine transformation:
  - we can find a similar at-least-1-dimensional family of sets
  - corresponding to each ellipsoid.

40. Proof (cont-d)

- Thus:
  - the dimension of the set of all sets S_c
  - is at least one larger than the dimension of the family of all ellipsoids,
  - i.e. at least $\dfrac{N \cdot (N+3)}{2} + 1 = r_{\min} + 2$.
- However, we have a $r_{\min}$-dimensional class of 1-dimensional families of sets.
- So the overall dimension of the set of all the sets S_c cannot be larger than $r_{\min} + 1$.
- This contradiction shows that the set S_c cannot be different from the enclosing minimal-volume ellipsoid.
- Thus, indeed, each set from each family from the optimal class is an ellipsoid.

41. Completing the Proof

- To complete the proof, we need to prove that ellipsoids in each family are concentric and homothetic.
- We have proven that each ellipsoid appears as an appropriate smallest-volume set.
- We know that each set S_c coincides with its smallest-volume enclosure.
- So, each ellipsoid appears as one of the sets S_c from one of the families from the optimal class.
- Let us again consider the unit ball centered at 0:
  - if the 1-dimensional family F_0 containing this ball is not invariant with respect to all possible rotations,
  - then we have at least a 1-dimensional group of different families containing the same ellipsoid.

42. Completing the Proof (cont-d)

- We have:
  - an $r_{\min}$-dimensional class of 1-dimensional families
  - covering the whole $(r_{\max}+1)$-dimensional family of ellipsoids.
- Thus, all elements of all families are different.
- So we cannot have several families containing the same ellipsoid.
- This argument shows that the family F_0 containing the unit ball *should be* rotation-invariant.
- All the sets from this family are included in each other and thus, cannot be rotated into each other.
- This means that each ellipsoid from this family F_0 must be rotation-invariant.

43. Completing the Proof (cont-d)

- This means that each ellipsoid from this family must be a ball concentric with our selected unit ball.
- Thus, it be homothetic to the original ball.
- For any other family F:
  - by selecting any ellipsoid E from this family and
  - by applying the affine transformation that transforms the above unit ball into E,
  - we get a new family $T(F_0)$ of concentric homothetic ellipsoids.
- An ellipsoid can only belong to one family.
- We thus conclude that the family F also consists of concentric homothetic ellipsoids.
- The result is proven.

44. Conclusions

- Wood is one the oldest construction materials; however:
  - in spite of several thousand years of experience with wooden constructions,
  - predicting and estimating mechanical properties of wooden constructions remains a difficult problem.
- One of the main reasons for this difficulty is that:
  - in contrast to many other constructions materials which are largely homogeneous and isotropic,
  - wood is highly inhomogeneous and anisotropic.
- Recently, a new property of wooden materials was discovered.
- It has a potential to make mechanical analysis of wooden structures more efficient.

45. Conclusions (cont-d)

- Namely, for wood:
  - iso-correlation surfaces (i.e., surfaces of equal correlation)
  - are well-approximated by concentric homothetic ellipsoids.
- The problem is that this property is purely empirical.
- It has no theoretical explanation and thus, engineers are understandably reluctant to rely on it.
- In this talk, we provide a theoretical explanation for this empirical fact.
- Thus, we make this property more reliable and therefore more useable.

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