INSTITUTE FOR RISK AND RELIABILITY

Leibniz Universität Hannover

Elucidating appealing features of differentiable autocorrelation functions: a study on the modified exponential kernel

M. Faes, M. Broggi, P.D. Spanos, M. Beer

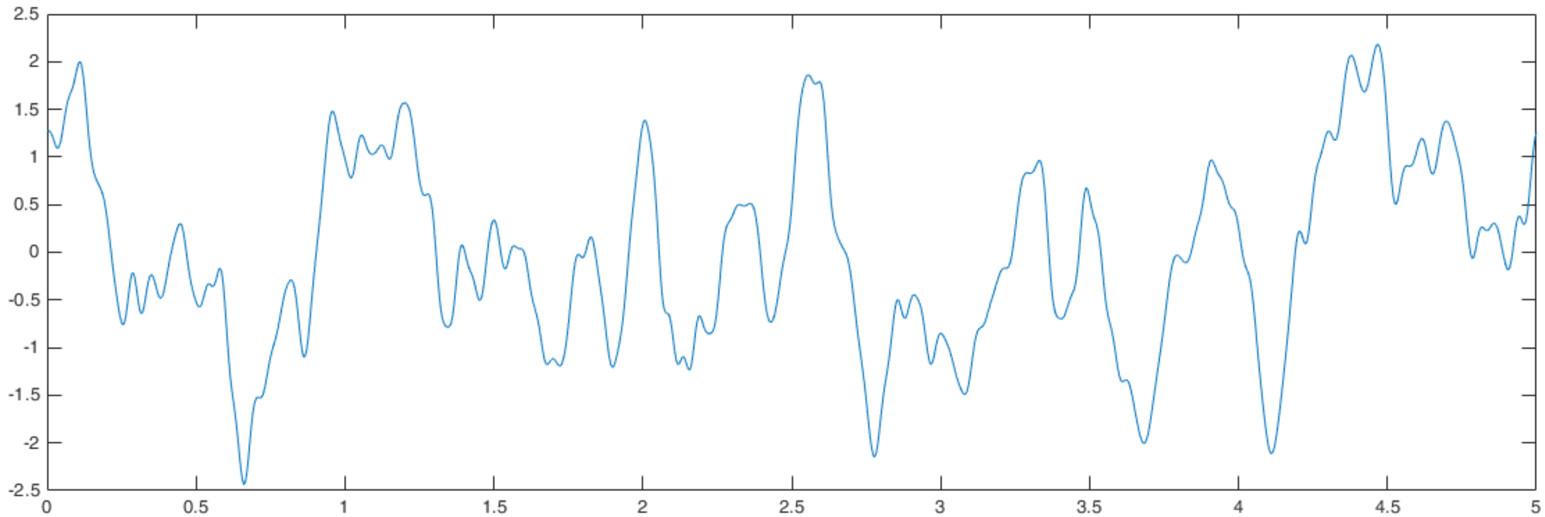
A small quiz

- Model a 1D spatial random field
  - Domain: [0m, 5m]
  - Exponential correlation
  - Correlation length: 0.05m
  - Standard deviation: 1m
  - Samples generated with spectral representation
  - Cut off frequency at 99% of energy of the signal
- You will be shown 2 samples
  - Which sample is more realistic?
  - Which sample is actually from an exponential correlation?

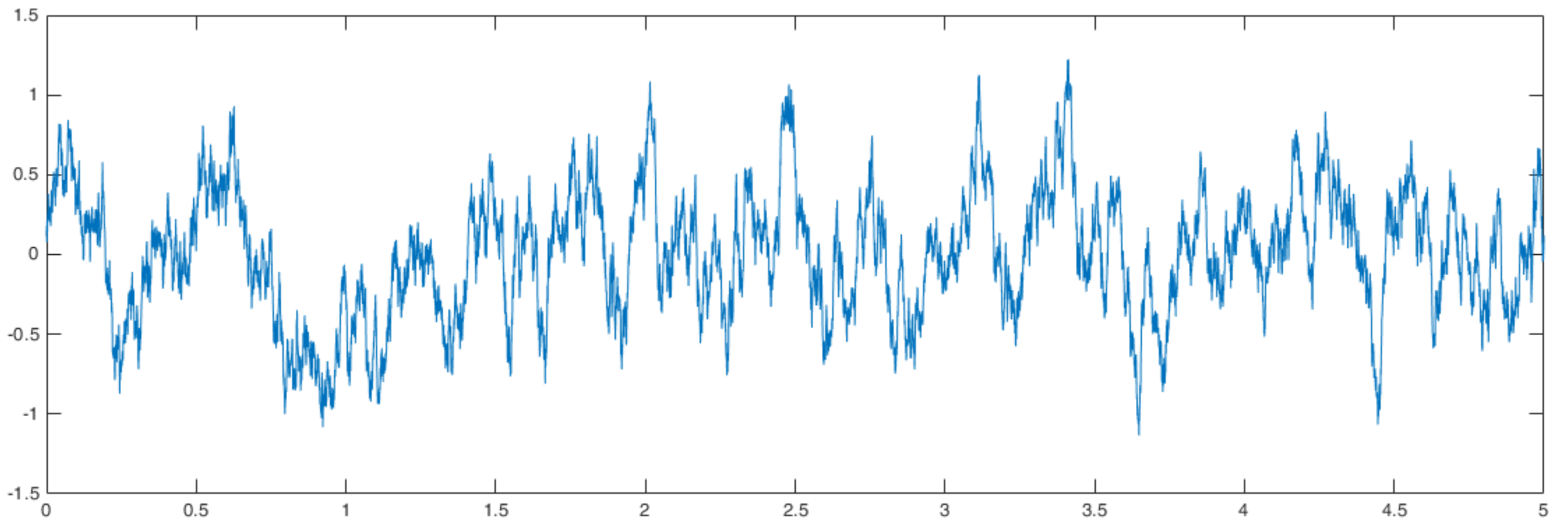
INSTITUTE FOR RISK AND RELIABILITY

Leibniz Universität Hannover

A small quiz



A



B

A small quiz

- If you think samples A is more realistic for modelling a spatially variating engineering quantity…
- That sample is not from an exponential correlation

Outline

- Introduction
- Comparison of autocorrelation function
- Convergence of series terms
  - Spectral representation
  - Karhunen–Loève expansion
- Conclusion

Introduction

- Most commonly used autocorrelation functions in random field modelling
  1. Exponential
  2. Squared exponential
  3. Modified exponential
- Modified exponential advantages*
  - differentiable everywhere
  - reduced Number of Terms N in the Expansion
  - better convergence

* P. Spanos, M. Beer and J. Red-Horse, Karhunen-Loeve Expansion of Stochastic Processes with a Modified Exponential Covariance Kernel. *Journal of Engineering Mechanics*,133, 2007.

Introduction
Modified Exponential Autocorrelation

- Traditional exponential correlation model

$$C(x_1, x_2) = e^{-|x_1 - x_2|/b} = e^{-|\tau|/b}$$

  - Non differentiable at the origin $\tau = 0$
  - Corresponding to a first-order AR random series in time*
    - $Y(x+1) = kY(x) + W(x)$
  - Directionality of the parameter x

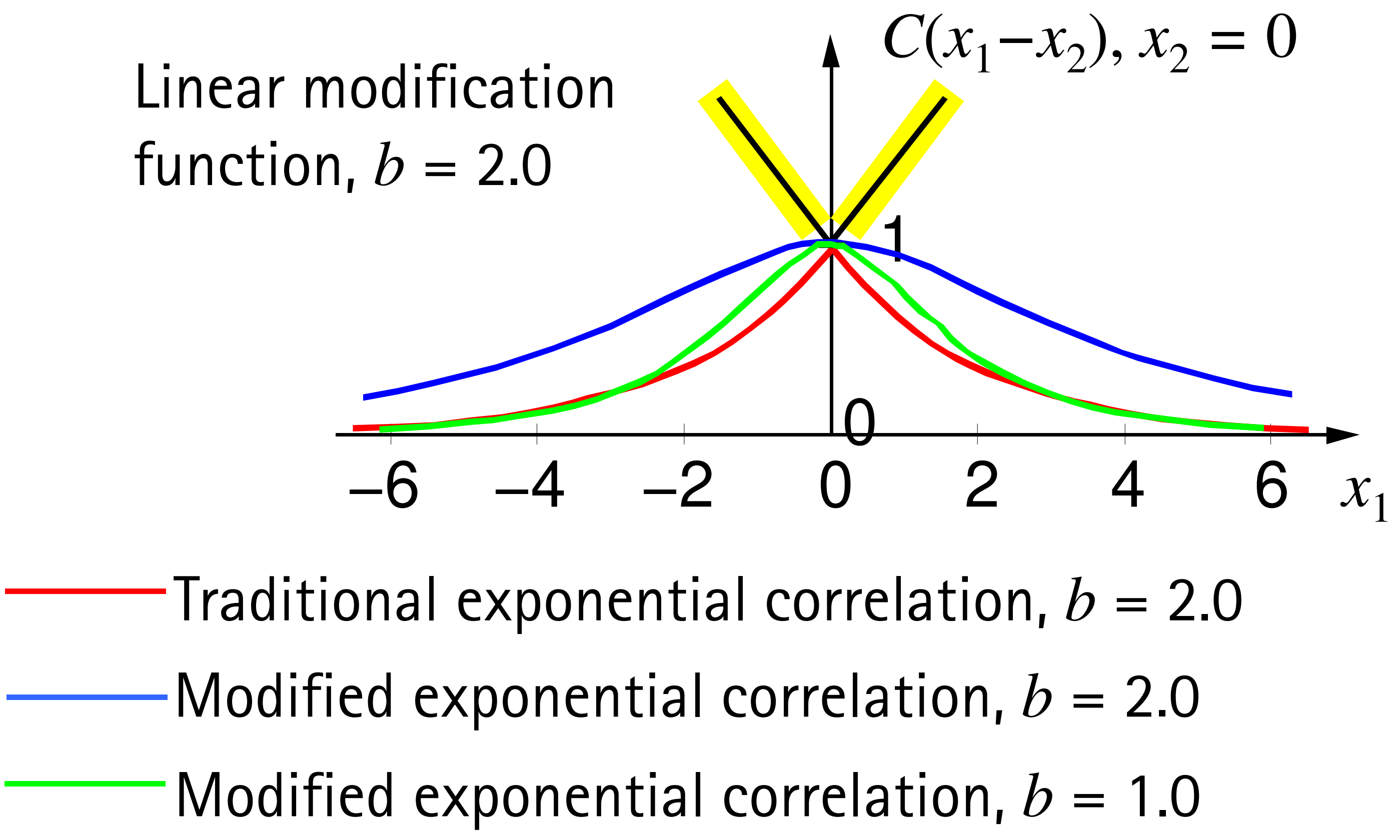
- Modified exponential correlation model

$$C(x_1, x_2) = e^{-\frac{|\tau|}{b}}(1 + |\tau|/b)$$

  - Differentiable at the origin $\tau = 0, \frac{\partial C}{\partial \tau} = 0$
  - Corresponding to a first-order AR random series in space*
    - $Y(x) = k[Y(x+1) + Y(x-1)] + W(x)$
  - Non-directionality of the parameter x

*(Vanmarcke, 1983)

Introduction
Modified Exponential Autocorrelation



Comparison of kernels
Autocorrelation functions and power spectra

- Autocorrelation function $R_{ff}(\tau)$ and power spectrum $S_{ff}(\omega)$ are linked through Wiener-Khintchine transform
- Considered autocorrelations:
  - Exponential kernel:

$$R_{ff}^{s}(\tau) = \exp(-|\tau|/b) \;\Leftrightarrow\; S_{ff}^{s}(\omega) = \frac{1}{\pi}\frac{b}{b^2\omega^2+1}$$

  - Modified exponential kernel

$$R_{ff}^{m}(\tau) = \exp(-|\tau|/b)(1+|\tau|/b) \Leftrightarrow S_{ff}^{m}(\omega) = \frac{1}{\pi}\frac{2b}{(b^2\omega^2+1)^2}$$

  - Squared exponential kernel

$$R_{ff}^{sq}(\tau) = \exp(-\tau^2/b^2) \Leftrightarrow S_{ff}^{sq}(\omega) = \frac{b}{2\sqrt{\pi}}\exp\left(-\frac{b^2\omega^2}{4}\right)$$

INSTITUTE FOR RISK AND RELIABILITY

Leibniz Universität Hannover

Comparison of kernels: spectral methods
Converge in energy

- Comparison based on convergence of energy content of realisations
- According to Shinozuka*, cut-off ω_u is determined as:

$$\int_0^{\omega_u} S_{ff}(\omega_n) d\omega = (1 - e_S) \int_0^{\infty} S_{ff}(\omega_n) d\omega$$

- The error convergence is computed as:

$$e_S(\omega_u) = 1 - \frac{\int_0^{\omega_u} S_{ff}(\omega_n) d\omega}{\int_0^{\infty} S_{ff}(\omega_n) d\omega}$$

- Integrals are solved analytically for these kernels

* Shinozuka, M., & Deodatis, G. (1991). Simulation of stochastic processes by spectral representation. *Journal of Engineering Mechanics*, *44*(4).

Comparison of kernels: KL expansion
Converge in mean error variance

- Comparison based on convergence of mean error variance
- Mean error variance e_σ of stochastic process over domain Ω is*:

$$1 - \frac{1}{|\Omega|}\sum_{i=1}^{n_{KL}} \lambda_i \leq e_\sigma$$

- Eigenvalues λ_i of $R_{ff}(\tau)$ determined from:

$$\int_\Omega R_{ff}(\tau)\psi_i(t')dt' = \lambda_i \psi_i(t)$$

* Betz, W., Papaioannou, I., & Straub, D. (2014). Numerical methods for the discretization of random fields by means of the Karhunen-Loève expansion. *Computer Methods in Applied Mechanics and Engineering*, *271*, 109–129.

Convergence in energy – Spectral Representation

- Traditional exponential

$$e_S = 1 - \frac{2}{\pi}\tan^{-1}(b\omega_u)$$

- Modified exponential

$$e_S = 1 - \frac{2}{\pi}\left(\frac{b\omega_u}{b^2\omega_u^2 + 1} + \tan^{-1}(b\omega_u)\right)$$

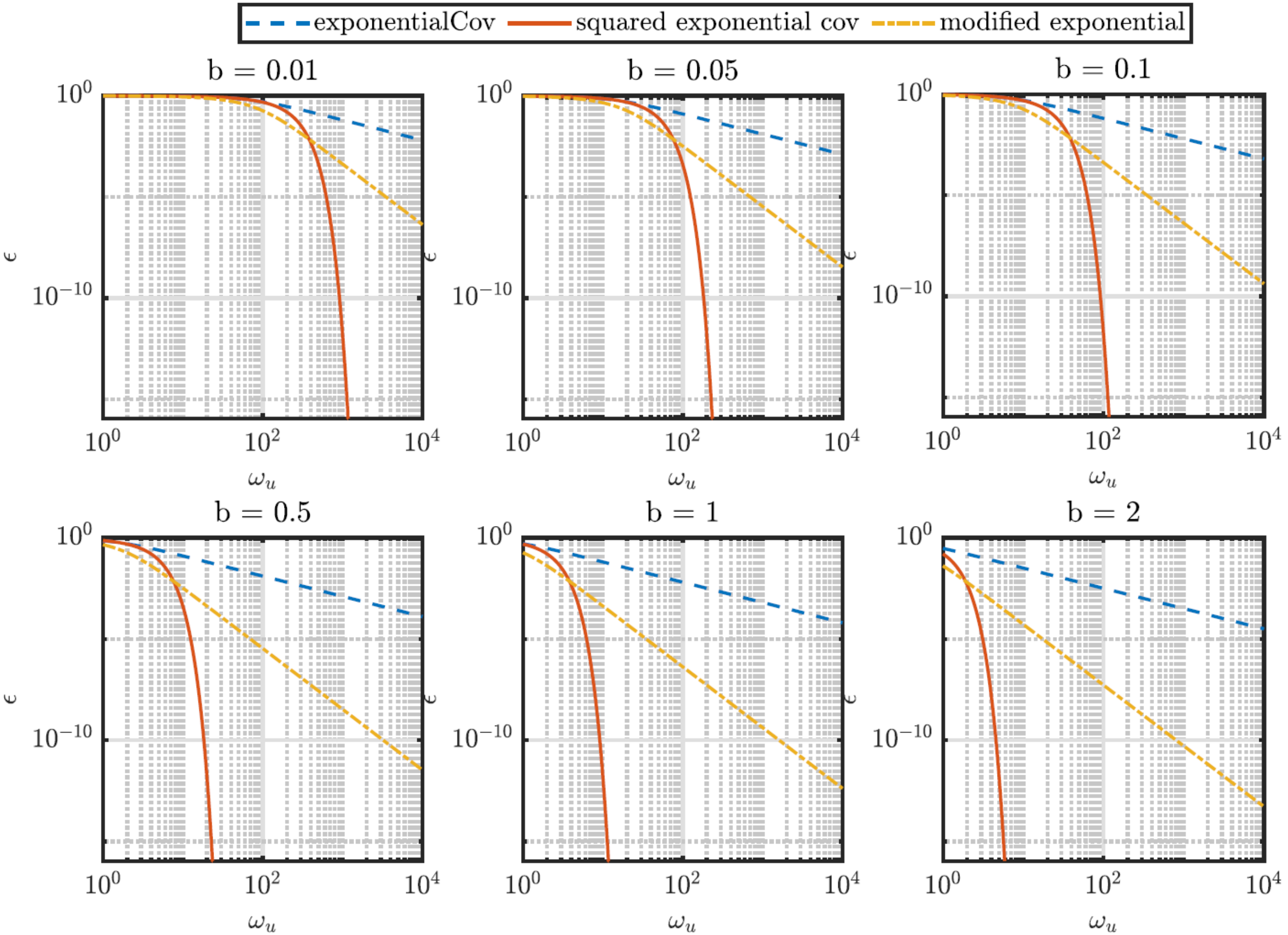
- Squared exponential

$$e_S = 1 - \operatorname{erf}\left(\omega_u \frac{b}{2}\right)$$

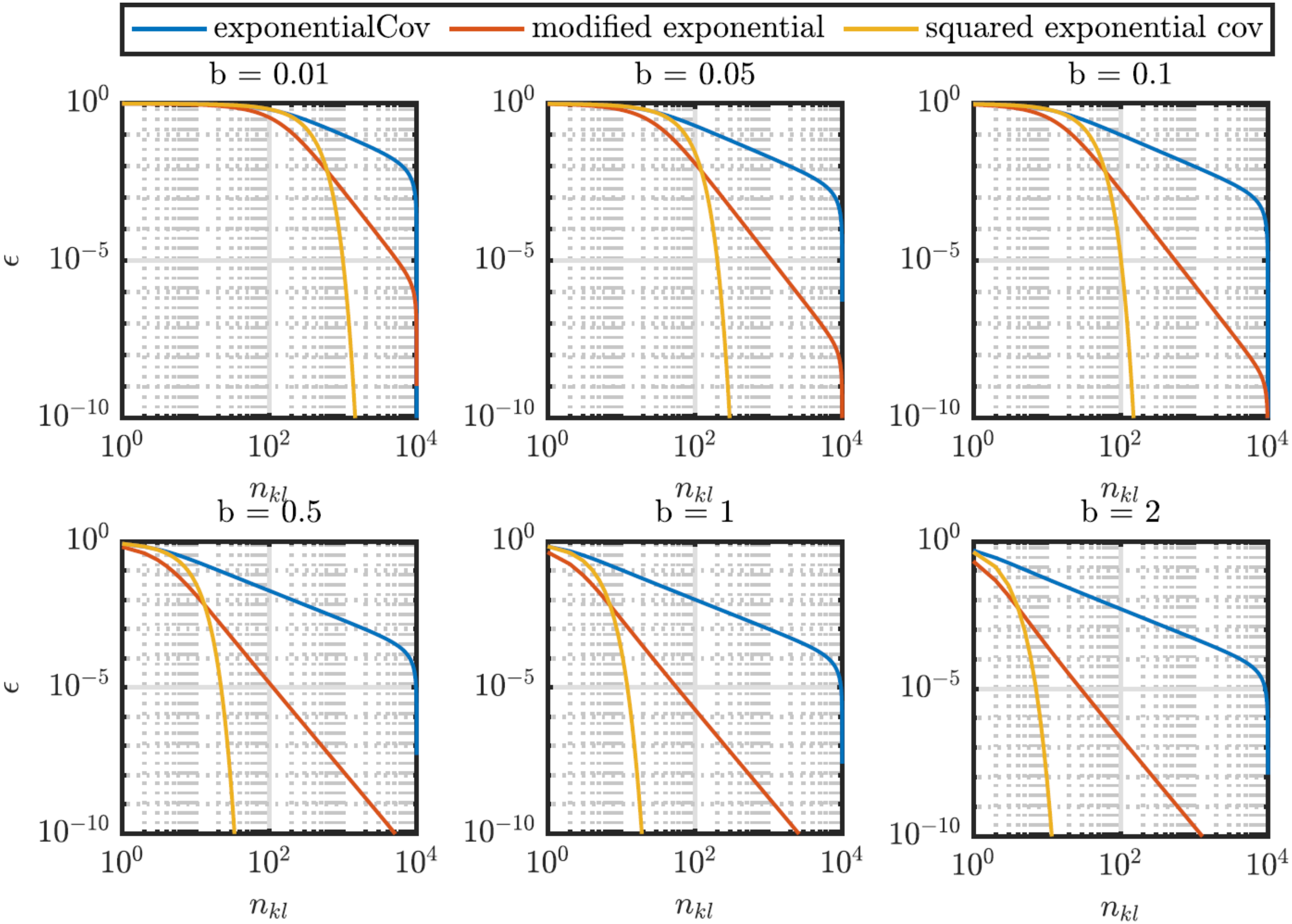
INSTITUTE FOR RISK AND RELIABILITY

Leibniz Universität Hannover

Convergence in energy – Spectral Representation



Convergence in MEV – Karhunen-Loeve expansion



Conclusions

- Don't use exponential autocorrelation for spatial random fields
- The number of stochastic components required to represent a stochastic process with the single exponential kernel is considerably larger when compared to a modified or squared exponential kernel
- True for both spectral representation and KL
- Superior convergence and more realistic samples from modified exponential, with characteristic desirable for engineering application