Guaranteed Minimization of the Bit Error Ratio for Correlated MIMO Systems

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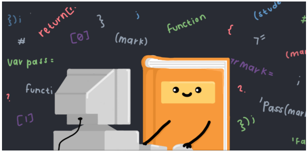
Guaranteed Results with Result Verification

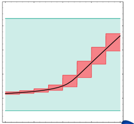
VERIFICATION — Are we building the product right? [Boehm]

FORMAL V
($\rightarrow$ model checking)

CODE V
($\rightarrow$ literate programming)



RESULT V
($\rightarrow$ interval arithmetic)



Principle: Prove mathematically the correctness of the computer result (fixed point theorems + set-based arithmetics)

Advantages: Account for rounding or conversion errors; propagate epistemic uncertainty

Disadvantages: Possibly too pessimistic ($\rightsquigarrow$ overestimation)

Approaches: Interval, affine, Taylor model, ... based methods

Result Verification: Applications

Computer assisted proofs

Smale's 14th: Do the properties of the Lorenz attractor exhibit that of a strange attractor?

Answer: Yes, proved by W. Tucker in 2002 with intervals

Other application areas

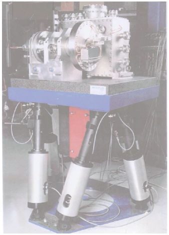
→ Computer graphics [Luther,Stolfi]
→ Finance/decision-making [Hu,Tsao]
→ Imprecise probability [Kreinovich,Ferson]

Main area: Engineering

→ Robotics [Jaulin,Merlet]
→ Chemical engineering [Stadtherr]
→ Particle accelerators [Makino,Berz]
→ Control theory [Walter,Rauh]
→ ... many more ...

And now: Wireless communications!

http://www.cs.utep.edu/interval-comp/



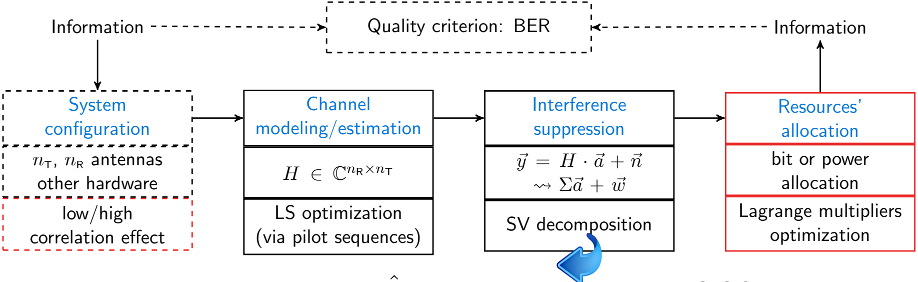
A positioner for the ESRF, Merlet

Focus: MIMO (**M**ultiple **I**nput **M**ultiple **O**utput) Systems

Improve the channel capacity/integrity without increasing the channel bandwidth or the transmit power

Method:	Multiple data streams are transmitted on the same frequency band and at the same time
Separation:	Spatial, for example, multiple antennas at the transmitter and receiver side at different locations
Correlation effect:	Caused by the proximity of the multiple antennas; transmit-to-receive paths might become too similar!
Channel capacity:	The information theoretic limit on the bit ratio (BER)
BER:	The number of bits per second that can be transmitted through a physical channel error free

Modeling and Simulation of a MIMO Digital Channel



Interference suppression $\to$ $\hat{L}$ independent, weighted SISO links

(pre: $\vec{z} = V\vec{a}$, post: $\vec{u} := U^\dagger \vec{z} = U^\dagger \left(U\Sigma V^\dagger\right) V\vec{a} + U^\dagger \vec{n} = \Sigma\vec{a} + \vec{w}$)

Each stage might be affected by uncertainty and numerical errors!

Uncertain factors: $\lambda_l = \sqrt{\xi_l}$ — singular values of H

σ^2 — the noise variance at the receiver side

Resources allocation: L number of activated layers, $P_s^{(l)}$ transmit power,

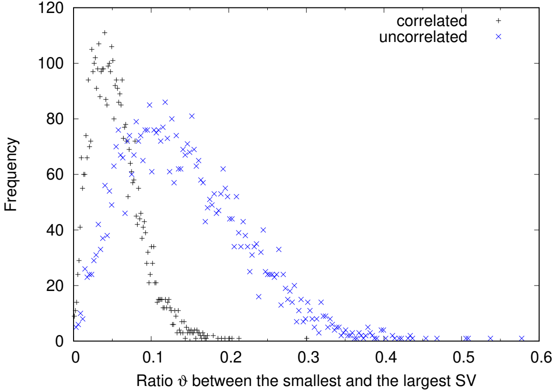
M_l constellation size

Focus: Compare (Un-)Correlated Channel Realizations

Weights λ_l are not equal

Proximity might make this stronger! Indicator: $\vartheta = \dfrac{\text{the smallest } \lambda}{\text{the largest } \lambda}$

Illustration for a (4×4) MIMO channel (5000 realizations each):



Enclosing the BER for Uncertain Parameters

BER $$P_b = \frac{2}{\sum_{l=1}^{L} \log_2 M_l} \sum_{l=1}^{L} \left(1 - \frac{1}{\sqrt{M_l}}\right) \cdot \mathrm{erfc}\left(\frac{\lambda_l}{2\sigma}\sqrt{\frac{3 \cdot P_s}{L(M_l - 1)}}\right)$$

Task: Minimize the BER for uncertain $\lambda_l \in [\underline{\lambda}_l, \overline{\lambda}_l]$, $\sigma \in [\underline{\sigma}, \overline{\sigma}]$

$\rightarrow$ Minimize the upper bound!

Bound: $$\frac{2}{\sum_{l=1}^{L} \log_2 M_l} \sum_{l=1}^{L} \left(1 - \frac{1}{\sqrt{M_l}}\right) \cdot \mathrm{erfc}\left(\frac{\underline{\lambda}_l}{2\overline{\sigma}}\sqrt{\frac{3P_s}{L(M_l - 1)}}\right)$$

Minimize wrt. $P_s^{(l)}$ ($\rightsquigarrow$ power allocation) and L, M_l ($\rightsquigarrow$ bit allocation)

Power allocation: Largange multipliers + software with result verification

Bit allocation: Non-linear mixed-integer programming problem + software with result verification + power allocation

Power allocation: Problem Formulation

Idea: Assign more power to the layers with small weights! (L, M_l fixed)

$\rightarrow$ $P_s^{(l)} = \frac{P_s}{L}$ (equally distributed) $\rightsquigarrow \pi_l^2 \cdot P_s^{(l)}$ so that $\sum_{l=1}^{L} \pi_l^2 \cdot P_s^{(l)} = P_s$

Method: Constrained optimization with Lagrange multipliers

$$J(\pi_1 \ldots \pi_L, \mu) = \frac{2}{\sum_{l=1}^{L} \log_2 M_l} \sum_{l=1}^{L} \left(1 - \frac{1}{\sqrt{M_l}}\right) \cdot$$

$$\operatorname{erfc}\left(\frac{\pi_l \lambda_l}{2\sigma} \sqrt{\frac{3 \cdot P_s}{L(M_l - 1)}}\right) + \mu \left(\overbrace{\sum_{l=1}^{L} \pi_l^2 - L}^{\text{constraint}} \right) \xrightarrow[\pi_l, \mu]{} \min$$

Power allocation: Verified Solution

Possibility 1 Mix analytical and numerical techniques

Stationary points: From the nonlinear algebraic system

$$\frac{\partial J(\pi_1 \ldots \pi_L, \mu)}{\partial \pi_l} = -\frac{2k_l}{\sqrt{\pi_l}} \left(c_l \lambda_l e^{-c_l^2 \lambda_l^2 \pi_l^2} \right) + 2\mu\pi_l = 0, \quad \sum_{l=1}^{L} \pi_l^2 - L = 0$$

with $k_l = \frac{2}{\sum_{l=1}^{L} \log_2 M_l} \cdot \left(1 - \frac{1}{\sqrt{M_l}}\right)$, $c_l = \frac{1}{2\sigma}\sqrt{\frac{3 \cdot P_s}{L(M_l - 1)}}$

Solve using software with result verification $\rightsquigarrow$

C-XSC `www2.math.uni-wuppertal.de/wrswt/xsc/cxsc.html`

The (bordered) Hessian can be shown to be built in such a way that a stationary point is a local minimum!

Possibility 2 Use global optimization directly (e.g., in C-XSC)

Possibility 2 is usually more afflicted by overestimation!

Bit allocation

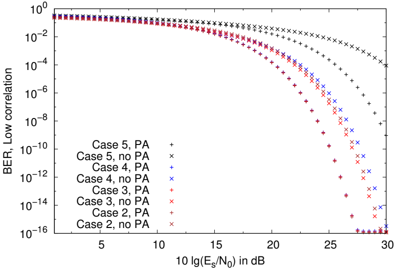
BER: $$P_b = \frac{2}{\sum_{l=1}^{L} \log_2 M_l} \sum_{l=1}^{L} \left(1 - \frac{1}{\sqrt{M_l}}\right) \cdot \mathrm{erfc}\left(\frac{\lambda_l}{2\sigma}\sqrt{\frac{3 \cdot P_s}{L(M_l - 1)}}\right)$$

→ M_l influences the overall BER

→ But also L (given throughput T)

case	L	layer 1	2	3	4
1	$L = 1$	256	—	—	—
2	$L = 2$	64	4	—	—
3	$L = 2$	16	16	—	—
4	$L = 3$	16	4	4	—
5	$L = 4$	4	4	4	4

(throughput $T = 8$ bit/s/Hz)



Constraint: $\sum_{l=1}^{L} \log_2 M_l = T$

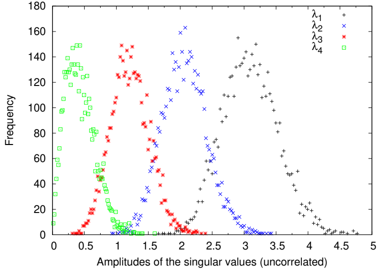
Solution: Brute force for small T and $\hat{L}$ possible (+ power allocation)!

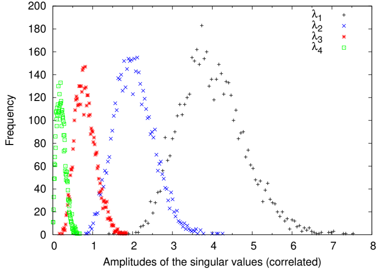
MIMO Link with Four Antennas

Simulation Settings

MIMO: Frequency flat, $n_\mathsf{T} = n_\mathsf{R} = 4$, $T = 8$ bit/s/Hz, $P_s = 1$ W

Two data sets with 5000 channel realizations each for correlated and uncorrelated case (simulated)



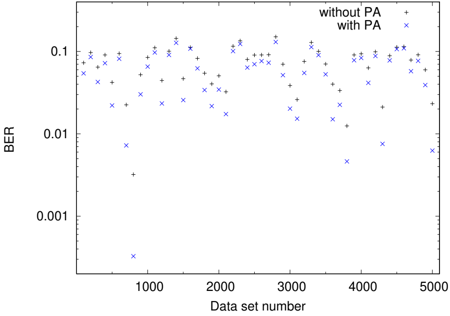


Results for the SNR 10 dB (corresponding to $\sigma \approx 0.2236$)

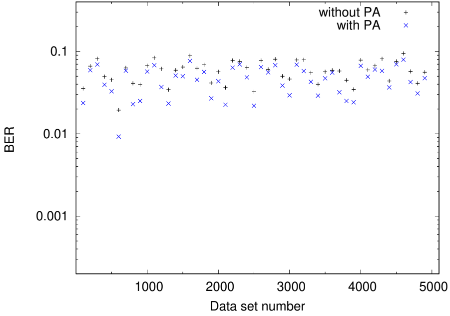
Power Allocation for Four Active Layers $(L = 4)$

Case $M_1 = 4,\ M_2 = 4,\ M_3 = 4,\ M_4 = 4$

Uncorrelated data set



Correlated data set



BER is reduced for each constellation of sigular values!

Bit Allocation for a (4×4) MIMO System

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \longrightarrow M_1 \geq M_2 \geq M_3 \geq M_4$$

([lowest, highest] BER for 5000 channel realizations)

	Layer M_1, M_2, M_3, M_4	BER (correlated)	BER-PA (correlated)	BER (uncorrelated)	BER-PA (uncorrelated)
			One active layer		
1	256, 0, 0, 0	[0.0023,0.1492]	the same	[0.0240, 0.13423]	the same
			Two active layers		
2	128, 2, 0, 0	[0.0022,0.1449]	[0.0001,0.1220]	[0.0232, 0.1304]	[0.0059, 0.1036]
3	64, 4, 0, 0	[$55{\cdot}10^{-6}$,0.1103]	[$4{\cdot}10^{-6}$,0.0959]	[0.0044, 0.0928]	[0.0007, 0.0749]
4	32, 8, 0, 0	[$10{\cdot}10^{-7}$,0.0808]	[$2{\cdot}10^{-7}$,0.0773]	[0.0002, 0.0599]	*[0.0001, 0.0556]*
5	16, 16, 0, 0	[$40{\cdot}10^{-7}$,0.1092]	[$3{\cdot}10^{-7}$,0.0981]	[$1.4{\cdot}10^{-4}$, 0.06710]	[$1.1{\cdot}10^{-4}$, 0.0589]
			Three active layers		
6	64, 2, 2, 0	[$8{\cdot}10^{-4}$,0.1279]	[$8{\cdot}10^{-6}$,0.1002]	[0.0127, 0.1121]	[0.0009, 0.0771]
7	32, 4, 2, 0	*[$12{\cdot}10^{-6}$,0.0926]*	*[$2{\cdot}10^{-6}$,0.0775]*	[0.0015, 0.0739]	[$6{\cdot}10^{-5}$, 0.0533]
8	16, 8, 2, 0	[$11{\cdot}10^{-6}$,0.1006]	[$5{\cdot}10^{-6}$,0.0936]	*[0.0001, 0.06417]*	[$2{\cdot}10^{-5}$, 0.0584]
9	16, 4, 4, 0	[$11{\cdot}10^{-5}$,0.1015]	[$1{\cdot}10^{-5}$,0.0972]	[$9{\cdot}10^{-5}$, 0.0850]	[$1{\cdot}10^{-5}$, 0.0785]
10	8, 8, 4, 0	[0.0001,0.1429]	[$7{\cdot}10^{-5}$,0.1282]	[$2{\cdot}10^{-5}$, 0.1048]	[$1{\cdot}10^{-5}$, 0.0916]
			Four active layers		
11	32, 2, 2, 2	[0.0106,0.1532]	[0.0032,0.1255]	[0.0073, 0.1426]	[0.0005, 0.1129]
12	16, 4, 2, 2	[0.0071,0.1252]	[0.0023,0.1181]	[0.0006, 0.1099]	[$7{\cdot}10^{-5}$, 0.1010]
13	8, 4, 4, 2	[0.0109,0.1665]	[0.0038,0.1529]	[$7{\cdot}10^{-5}$, 0.1419]	[$4{\cdot}10^{-5}$, 0.1344]
14	4, 4, 4, 4	[0.0414,0.2180]	[0.0228,0.2028]	[0.0014, 0.1909]	[0.0002, 0.1785]

All four layers should never be activated at the same time!

Conclusions

Results:

→ Problem solved by a mixed analytical/numerical technique with result verification

→ At least the weakest layer should be switched off

→ For correlated systems, resource allocation plays an especially important role

→ Best performance for two active layers

Future work:

→ Analyse the influence of the noise (σ)

→ Use GMD instead of SVD for obtaining equal weights — does the performance improve?

Thank you for your attention!