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Local Reliability Based Sensitivity Analysis with the Moving Particles Method

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Summary

- **Local reliability based sensitivity analysis:**
  - Dependence of the failure probability on model parameters

$$P_F = \int_{g(x)<0} p(x)\mathrm{d}x$$

  - Obtain partial derivatives of P_F with respect to the model parameters
  - **Objective: Less function evaluations than finite differences**

- **Solution:**
  - Score function approach, consider sensitivity w.r.t. parameters of the pdf p(x)
  - Estimation of the failure probability: moving particles method

- **No additional function evaluations are necessary; but: c.o.v. of the estimator increases due to the low number of particles**

  -> necessity for a multilevel approach

Introduction

- Subset Simulation

$$P_F = \prod_{i=1}^{M_T} P(F_i|F_{i-1}) = \prod_{i=1}^{M_T} P_i$$

- Moving Particles

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Moving Particles (A. Guyader et al., AMO 64, 2011)

- Maximum number of domains for subset simulation?

- Consider N_P initial samples X_i and compute $g_i = g(X_i)$
- Rank the samples according to the value g_i
- Move sample X_j with $g_j = \max g(X_i)$ (or the k samples with maximal g_i)
  - move: apply MCMC starting from one of the remaining samples
  - accept, if $g_i^* < g_i$

- The number of moves to get a sample into the failure domain follows a Poisson distribution with parameter $\lambda = -\log P_F$

- Count moves for each initial sample, estimate λ

$$\hat{\lambda} = \frac{\sum_{j=1}^{N} M_j}{N_P} \qquad \hat{P}_F = \exp(-\hat{\lambda}) = \left(\frac{N-1}{N}\right)^{M_T}$$

Comparison

	Subset Simulation	Particle Method
parallel implementation	+	+
number of moves per step	high	low
number of possible seeds	low	high
burn-in	+	-

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Local Reliability Analysis

- Starting point
$$\frac{\partial P_F}{\partial \theta} = P_F \sum_{i=1}^{M_T} \frac{1}{P_i} \frac{\partial P_i}{\partial \theta}$$

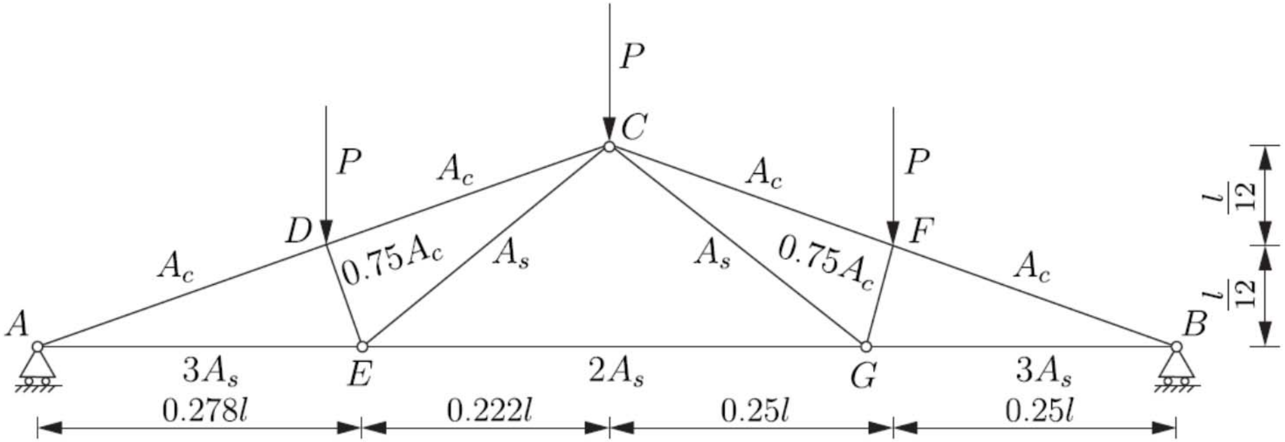
  with the conditional probabilities
$$P_i = \int I_{g<g_i}(x) p(x|G < g_{i-1}) \mathrm{d}x = \int I_{g<g_i}(x) \frac{I_{g<g_{i-1}}(x) p(x)}{P(G < g_{i-1})} \mathrm{d}x.$$

- Compute the partial derivatives of the conditional probabilities
$$\frac{\partial P_i}{\partial \theta} = E_{G<g_{i-1}}[I_{g<g_i}(x) \frac{\partial \ln p(x)}{\partial \theta}] - \sum_{j=1}^{i-1} \frac{1}{P_j} \frac{\partial P_j}{\partial \theta}$$

- Finally, as $\hat{P}_1 = \hat{P}_2 = ... = \hat{P}_{M_T}$:

$$\frac{\partial \hat{P}_F}{\partial \theta} = \left(\frac{N-1}{N} \right)^{M_T - 1} \left(\frac{1}{N} \sum_{j=1}^{N} I_{g<g_{M_T}}(x_j^{(M_T)}) \left. \frac{\partial \ln p(x)}{\partial \theta} \right|_{x = x_j^{(M_T)}} \right)$$

Example: Roof Truss Structure



$$g(q, \ell, A_C, E_C, A_S, E_S) = 0.03 - \frac{q\ell^2}{2}\left(\frac{3.81}{A_C E_C} + \frac{1.13}{A_S E_S}\right)$$

	q [N/m]	ℓ [m]	A_C [m^2]	E_C [N/m^2]	A_S [m^2]	E_S [N/m^2]
mean value	20000	12	0.04	2×10^{10}	9.82×10^{-4}	1×10^{11}
std. dev.	1400	0.12	0.0048	1.2×10^{9}	5.892×10^{-5}	6×10^{9}

Example: Roof Truss Structure

	P_F	$\frac{\partial P_F}{\partial m_q}$	$\frac{\partial P_F}{\partial m_\ell}$	$\frac{\partial P_F}{\partial m_{A_C}}$	$\frac{\partial P_F}{\partial m_{E_C}}$	$\frac{\partial P_F}{\partial m_{A_S}}$	$\frac{\partial P_F}{\partial m_{E_S}}$
estimate	0.00937	1.10e-5	0.0403	-2.110	-3.71e-12	-186	-1.81e-12
elasticity		23.5	51.9	-9.1	-8.0	- 19.6	-19.5
c.o.v.	0.05	0.05	0.08	-0.06	-0.08	-0.07	-0.06
rel. error [%]	0.5	0.7	0.9	1.0	1.2	0.1	0.6
		$\frac{\partial P_F}{\partial \sigma_q}$	$\frac{\partial P_F}{\partial \sigma_\ell}$	$\frac{\partial P_F}{\partial \sigma_{A_C}}$	$\frac{\partial P_F}{\partial \sigma_{E_C}}$	$\frac{\partial P_F}{\partial \sigma_{A_S}}$	$\frac{\partial P_F}{\partial \sigma_{E_S}}$
estimate		1.57e-5	0.0182	2.5047	1.93e-12	204	1.99e-12
elasticity		2.4	0.2	1.3	0.2	1.3	1.3
c.o.v.		0.06	0.22	0.08	0.20	0.09	0.09
rel. error [%]		2.9	1.4	1.3	3.4	0.8	1.0

Example: High-dimensional Problem

- Performance function

$$g(X) = \frac{1}{1000 + \sum_{i=1}^{100} X_i} - \frac{1}{1000 + 3\sqrt{100}},$$

- X_i: standard normal, correlation ρ

$\rho = 0$	P_F	$\frac{\partial P_F}{\partial m}$	$\frac{\partial P_F}{\sigma}$
estimate	1.35e-3	4.44e-2	1.32e-2
c.o.v.	0.069	0.069	0.083
rel. error [%]	0.3	0.1	1.0

$\rho = 0.5$	P_F	$\frac{\partial P_F}{\partial m}$	$\frac{\partial P_F}{\sigma}$
estimate	0.336	0.514	0.156
c.o.v.	0.024	0.027	0.70
rel. error [%]	0.1	0.1	1.3

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Multilevel extension

- Family of performance functions:

$$\{g_i(x)\},\ i = 0, ..., n,\ g_n(x) = g(x)$$

- Rewrite the expectation

$$E_{G<g_{M_T-1}}[I_{g<g_{M_T}} \frac{\partial \ln p(x)}{\partial \theta}]$$

as a telescoping sum

$$E_{G_0<g_{M_T-1}}[I_{g_0<g_{M_T}} \frac{\partial \ln p(x)}{\partial \theta}] + \sum_{i=1}^{n} E_{G_{i-1}<g_{M_T-1}}[I_{g_i<g_{M_T}} \frac{\partial \ln p(x)}{\partial \theta} - I_{g_{i-1}<g_{M_T}} \frac{\partial \ln p(x)}{\partial \theta}]$$

and estimate each expectation separately

Example: Burgers' equation

- Nonlinear SPDE $$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \alpha(x,\theta)\frac{\partial^2 u}{\partial x^2}$$
- KL expansion, viscosity: $$\alpha(x,\theta) = 1 + \sum_{i=1}^{M} \sqrt{\lambda_i} f_i(x)\xi_i(\theta)$$
  - Exponential covariance Kernel, standard normal random variables
- Polynomial chaos expansion: $$u(x,t,\xi) = \sum_{i=0}^{P} u_i(x,t)\psi_i(\xi)$$
- PDEs for the expansion coefficients:

$$\sum_{i=0}^{P} \frac{\partial u_i}{\partial t}\psi_i(\xi) + \sum_{i=0}^{P} u_i\psi_i(\xi) \sum_{j=0}^{P} \frac{\partial u_j}{\partial x}\psi_j = (1+\sum_{i=1}^{M} \sqrt{\lambda_i} f_i(x)\xi_i(\theta))(\sum_{i=0}^{P} \frac{\partial^2 u_i}{\partial x^2}\psi_i(\xi))$$

- Galerkin scheme, finite differences

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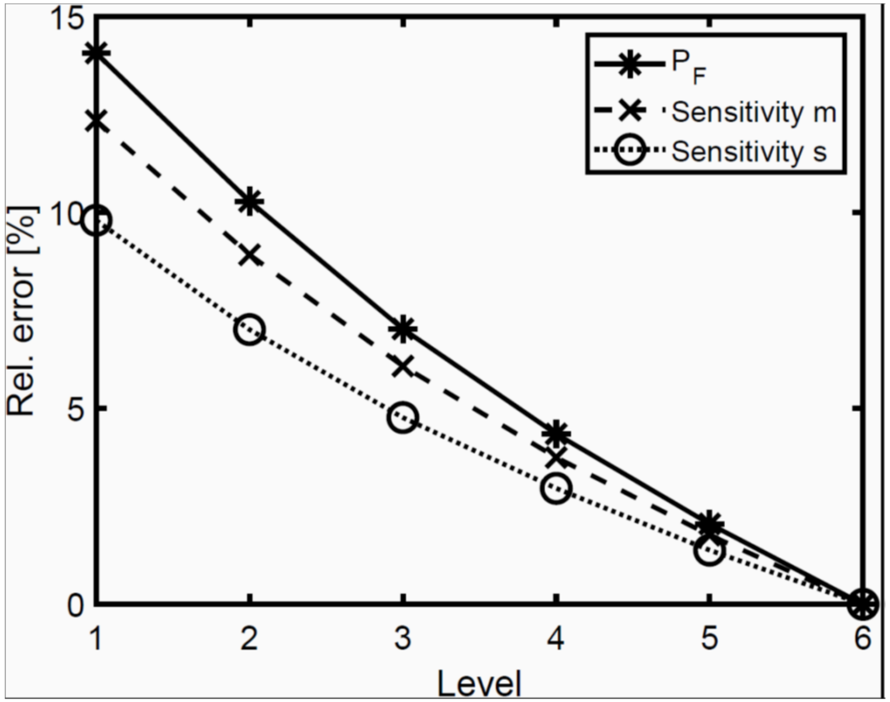
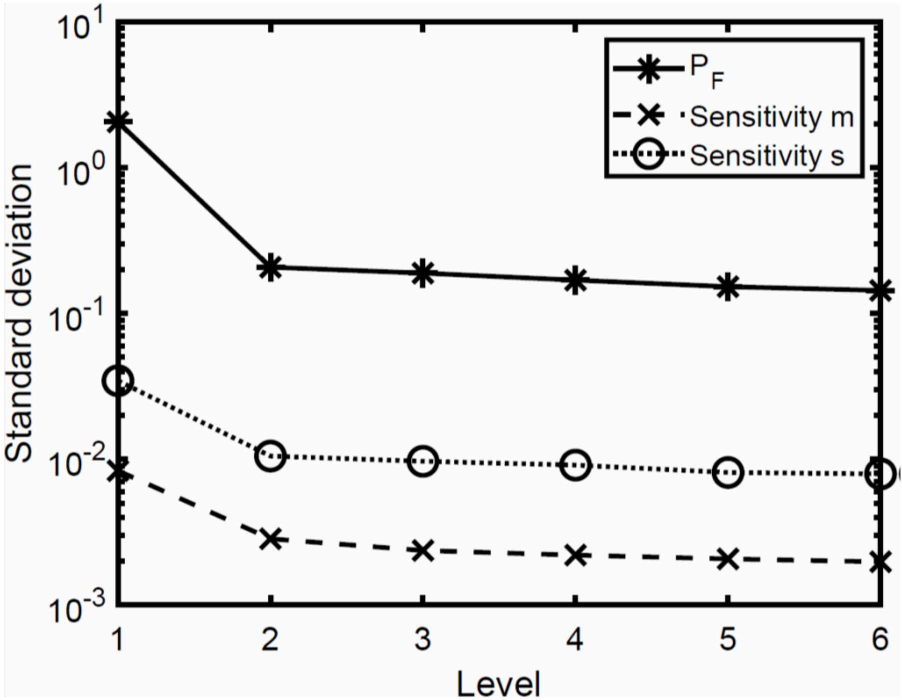
Example: Burgers' equation

$$\text{Failure: } u(x = 0.5, t = T) > 0.42$$





Conclusions

- A local reliability based sensitivity analysis based on the moving particles method has been introduced.
- The sensitivity analysis can be carried out in a single postprocessing step after the last particle has been moved into the failure domain.
- Due to the relatively low number of particles on which the estimator is based, the coefficient of variation should be further decreased.
- A multilevel method helps to reduce the standard deviation of the estimator and thus to improve the efficiency of the sensitivity analysis.
- Additional savings due to nested approximations (not considered here)

Acknowledgements

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Thank you very much for your attention!

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