9th International Workshop
on Reliable Engineering
Computing 2021

On the Solution of Forward and Inverse Problems in Possibilistic Uncertainty Quantification for Dynamical Systems

Why do we need possibilistic filters and
how can we design them?

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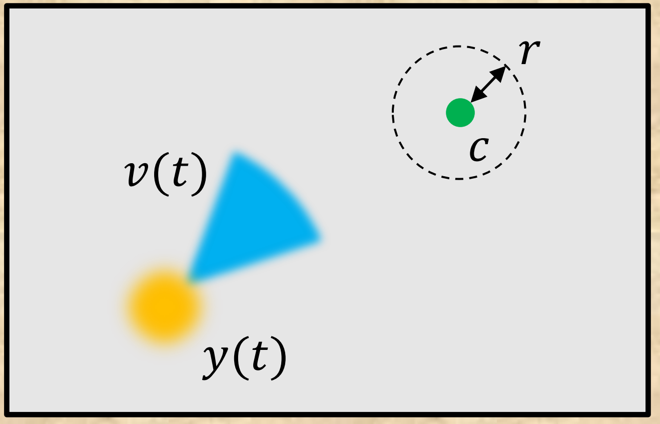
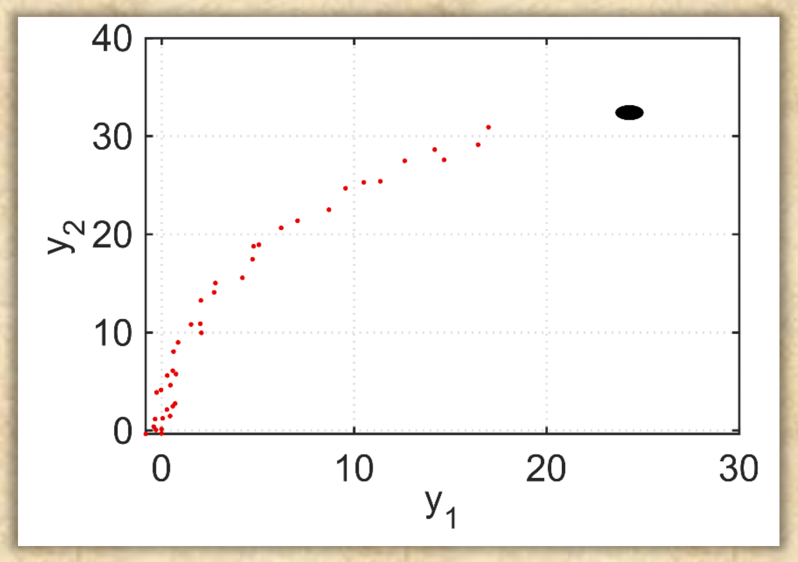
Dynamic Conjunction Analysis

Obstacle Avoidance



- collision: $\min_{t>0} \|y(t) - c\| \leq r$

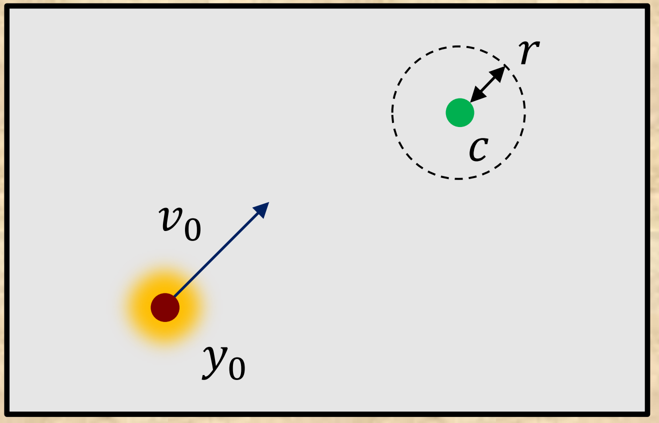


Uncertainties

- unknown initial position and velocity
  - vacuous
- system dynamics with process noise
  - epistemic
- sensor output with Gaussian measurement error
  - aleatory
- QoI: current and future states
  - mixed/polymorphic

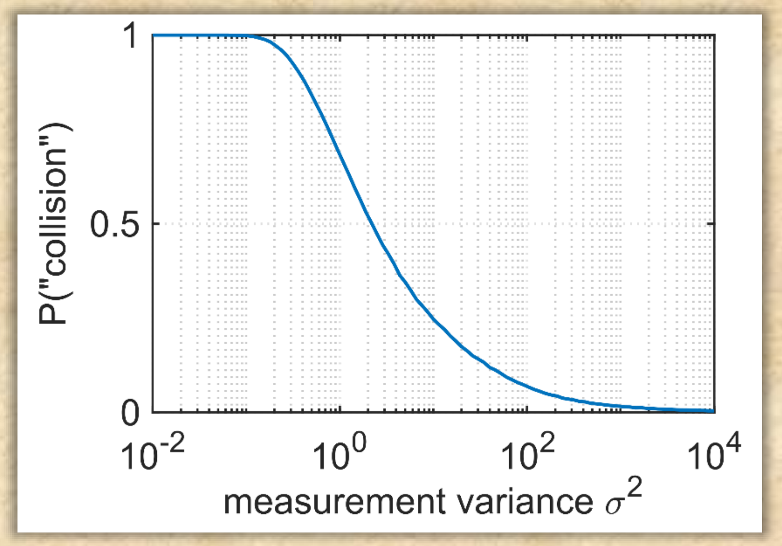
Static Conjunction Analysis

Obstacle Avoidance



- collision: $\min_{t>0} \|y_0 + v_0 t - c\| \leq r$
- known (constant) velocity v_0
- unknown initial position y_0
- Gaussian measurement error
  $$z = y_0 + \widetilde{W}$$

Bayesian Inference



- large measurement error yields high belief of avoidance (for most priors) – even if on collision course
  - ❖ False Confidence Theorem for *additive* belief measures

Balch, M. S., Martin, R., & Ferson, S. (2019). Satellite conjunction analysis and the false confidence theorem. Proceedings of the Royal Society A, 475(2227), 20180565.

Possibility Theory and Imprecise Probabilities

Axiomatic Approach

- (normal) elementary possibility function/membership function
  $$\pi : \Omega \to [0,1]$$

- possibility measure $\Pi(U) = \sup_{\omega \in U} \pi(\omega)$
  - complete
  - normal
  - positive
  - bounded
  - monotone
  - maxitive

$$\Pi(U_1 \cup U_2) = \max\big(\Pi(U_1), \Pi(U_2)\big)$$

- (dual) necessity $\mathrm{N}(U) = 1 - \Pi(U^{\mathrm{c}})$

Hose, D., & Hanss, M. (2021). A universal approach to imprecise probabilities in possibility theory. *International Journal of Approximate Reasoning*.

Imprecise Probabilities (IP)

- credal set of possibility measures $\mathbb{C}(\pi) = \{\mathrm{P} : \mathrm{P}(U) \leq \Pi(U)\ \forall\, U \in \Sigma\}$
- possibilities (necessities) are coherent upper (lower) previsions, capacities, monotone measures, …

IP-to-Π-Transform

- family of probabilities $\mathbb{P} = \{\mathrm{P}_1, \mathrm{P}_2, \dots\}$
  - find outer approximation $\mathbb{C}(\pi)$
- candidate elementary possibility function $\beta : \Omega \to [0,1]$

$$\pi(\omega) = \sup_{\mathrm{P} \in \mathbb{P}} \mathrm{P}(\{\xi \in \Omega : \beta(\xi) \leq \beta(\omega)\})$$

Imprecise Variables and Unknown Parameters

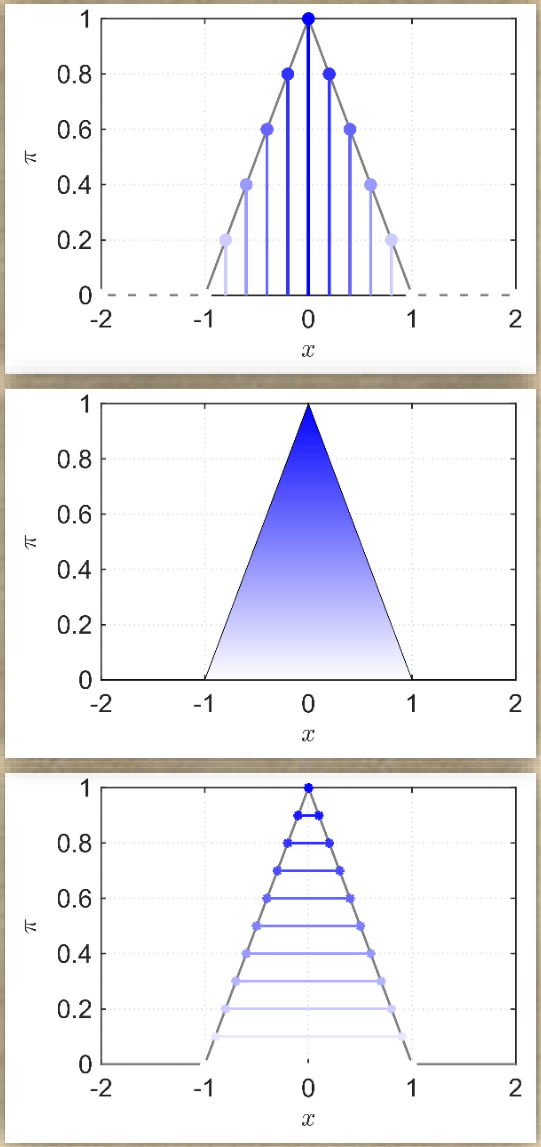
Same (Possibilistic) Calculus for…

… Predictive Distributions $\pi_{\tilde{X}} : \mathbb{X} \to [0,1]$

- imprecise (random) variable $\tilde{X}: \Omega \to \mathbb{X}$
- pointwise memberships are "imprecise likelihoods"
- α-cuts $K^{\alpha}_{\pi_{\tilde{X}}} = \{x \in \mathbb{X} : \pi_{\tilde{X}}(x) > \alpha\}$ are prediction sets
  - ❖ occurrence probability $\mathrm{P}\big(\tilde{X}(\omega) \in K^{\alpha}_{\pi_{\tilde{X}}}\big) \geq 1 - \alpha$

… Confidence Distributions $\gamma_{\Theta}: \mathbb{T} \to [0,1]$

- unknown parameter $\Theta \in \mathbb{T}$
- pointwise memberships are p-values of hypothesis that $\Theta = \theta$
- α-cuts $K^{\alpha}_{\gamma_{\Theta}} = \{\theta \in \mathbb{T} : \gamma_{\Theta}(\theta) > \alpha\}$ are confidence sets
  - ❖ coverage probability $\mathrm{P}\big(\Theta \in K^{\alpha}_{\gamma_{\Theta}}(\omega)\big) \geq 1 - \alpha$



Liu, C. & Martin, R. (2020). Inferential models and possibility measures. *Researchers.One*, 20.08.00004v1.

From IP to Confidence

Possibilistic Inference

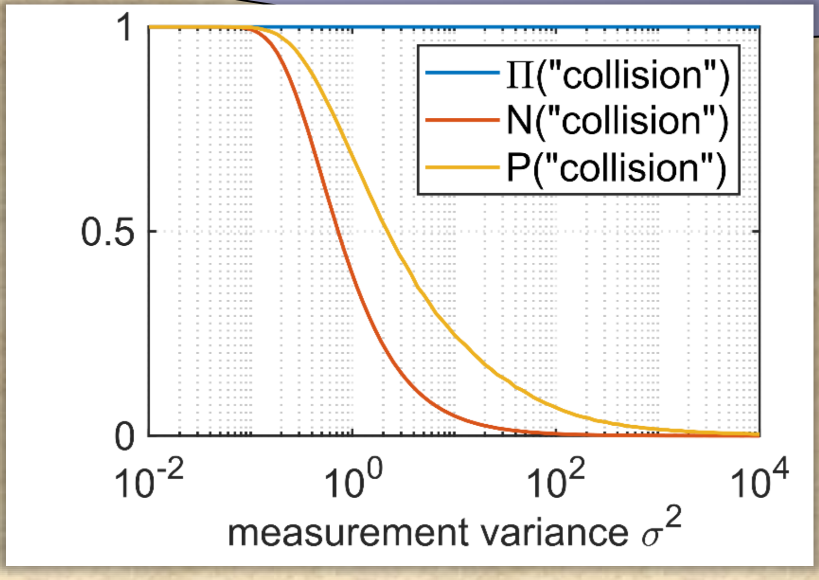
$$\gamma_{\Theta|\tilde{Q}=q}(\theta) = \sup_{u\,:\,q=\phi(\theta,u)} \pi_{\tilde{U}}(u)$$

- imprecise statistical model

$$\tilde{Q} = \phi(\Theta, \tilde{U})$$

  - imprecise pivot $\tilde{U} \sim \pi_{\tilde{U}}$
  - observation $\tilde{Q} = q$
  - unknown (+ nuisance) parameter Θ

Hose, D., & Hanss, M. (2021). A universal approach to imprecise probabilities in possibility theory. *International Journal of Approximate Reasoning*.



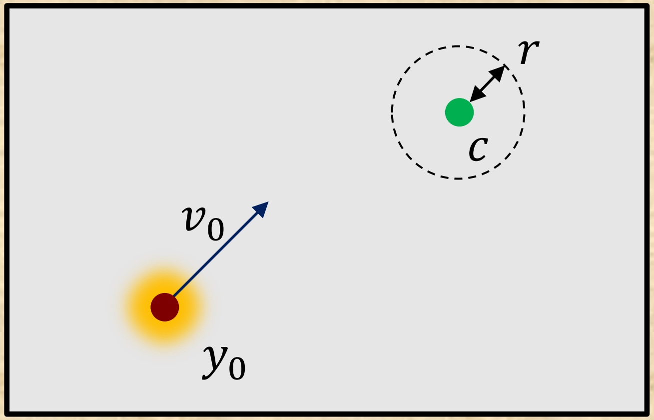
- no false confidence!

Static Conjunction Analysis

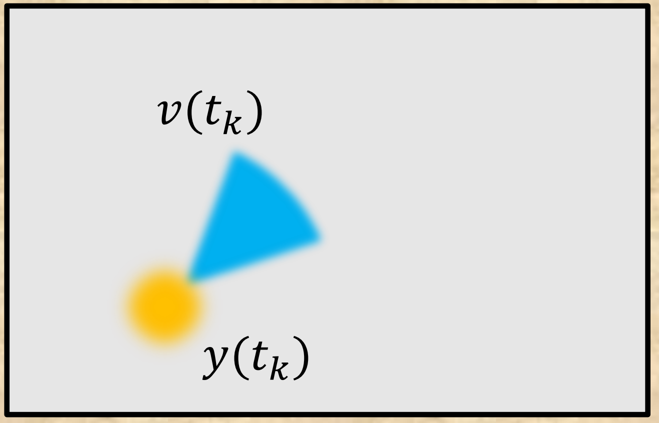
- known constant velocity v_0
- unknown initial position y_0
- Gaussian measurement error

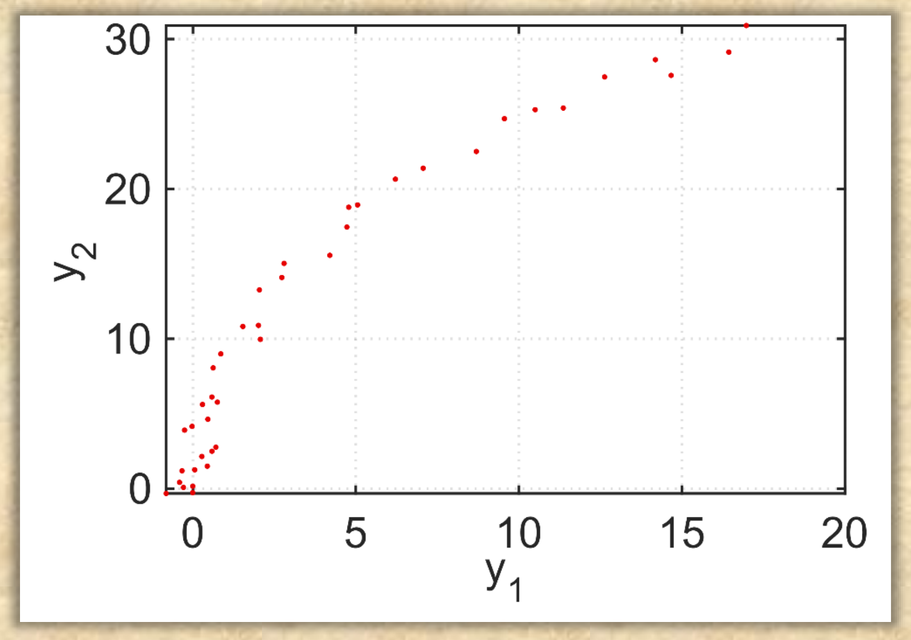
$$z = y_0 + \tilde{W}$$

- collision: $\min_{t>0} \|y_0 + v_0 t - c\| \leq r$



Dynamic Filtering





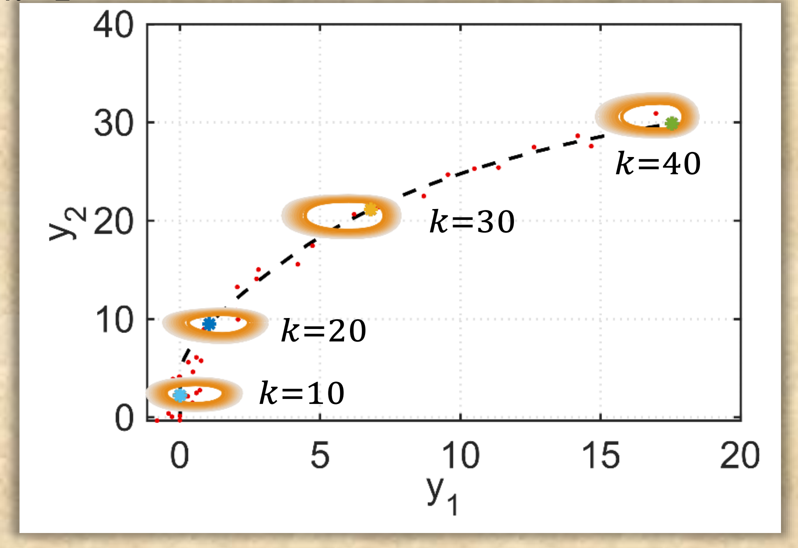
Statistical Model

- unknown dynamic states
  - $y_k = y(t_k), v_k = v(t_k)$
- sensor output with Gaussian measurement error $\widetilde{W}$ with covariance C
  - $z_k = y_k + \widetilde{W}_k$
- (time-discrete) LTI system with epistemic process noise $\widetilde{N}$
  - $y_{k+1} = y_k + \Delta t \cdot v_k$
  - $v_{k+1} = v_k + \widetilde{N}_k$
- unknown initial state $\widetilde{Y}_0, \widetilde{V}_0$

Possibilistic Filtering

- statistical model ϕ
  - system dynamics $\begin{cases} 0 = x_1 - f(x_0) - \widetilde{N}_0 \\ \vdots \\ 0 = x_k - f(x_{k-1}) - \widetilde{N}_{k-1} \end{cases}$
  - measurements $\begin{cases} y_1 = g(x_1) + \widetilde{W}_1 \\ \vdots \\ y_k = g(x_k) + \widetilde{W}_k \end{cases}$
  - initial conditions $x_0 = \tilde{X}_0$

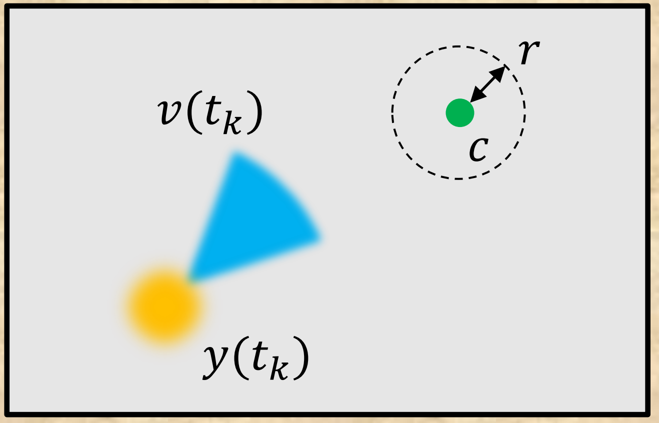
Feasibility Problem!
(convex for LTI systems)



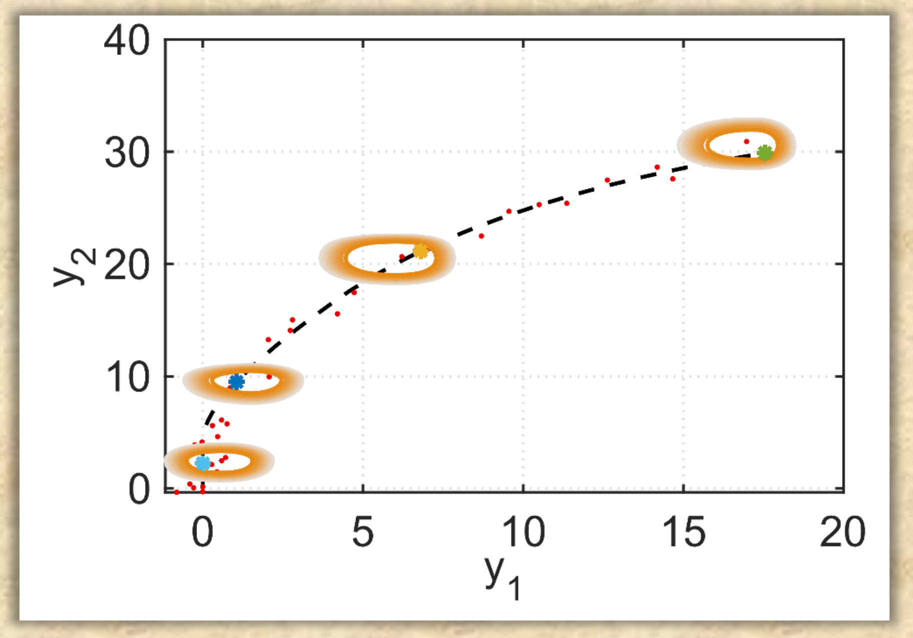
- imprecise pivots $\widetilde{U}$
  - $\pi_{\widetilde{W}_1,\dots,\widetilde{W}_k}(w_1,\dots,w_k) = 1 - F_{\chi^2_{2k}}\left(\sum_{j=1}^{k} w_j^{\mathrm{T}} \cdot \mathrm{C}^{-1} \cdot w_j\right)$ "aleatory"
  - $\pi_{\widetilde{N}_1}(n) = \cdots = \pi_{\widetilde{N}_{k-1}}(n) = \begin{cases} 1 & \text{if } |n| \le n_{\max} \\ 0 & \text{otherwise.} \end{cases}$ "epistemic"
  - $\pi_{\tilde{X}_0} \equiv 1$ "vacuous"

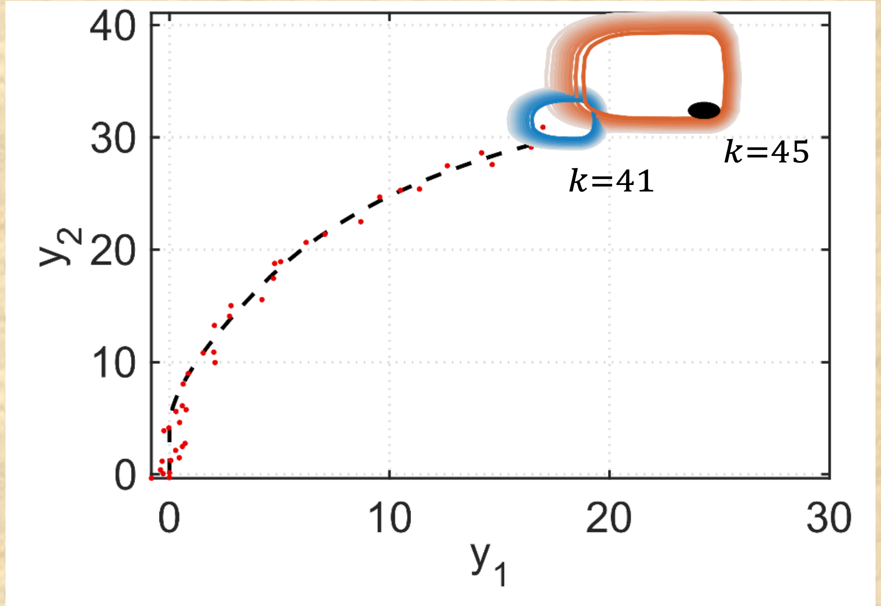
Obstacle Avoidance

Dynamic Conjunction Analysis (revisited)



- expand feasibility problem by predictions of future trajectories under process noise
- future measurement error irrelevant





Conclusion

Takeaway Messages

- (additive) Bayesian filters exhibit false confidence phenomena
- membership functions may be used to specify both predictive IP distributions and confidence distributions
  - same calculus
- Π-theory facilitates (subjective and frequentist) inference in imprecise(!) statistical models
- straight-forward filtering, prediction, smoothing, … formulation via mathematical programming
  - conceptually similar to moving horizon estimation

ISIPTA 2021

- recursive filtering formulation
  - initialization
  - prediction
  - update
  - (resampling)
- particle-based implementation

Hose, D., & Hanss, M.: A Recursive Formulation of Possibilistic Filters. *Proceedings of the Twelfth International Symposium on Imprecise Probabilities: Theories and Applications, 2021.*

Thank you for listening!

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