Interval analysis of the forced vibration of beams with uncertain damage

F. Cannizzaro, N. Impollonia, G. Cocuzza Avellino, S. Caddemi, and I. Caliò
Department of Civil Engineering and Architecture, University of Catania, Catania 95125, Italy,
{francesco.cannizzaro, nicola.impollonia, giuseppe.cocuzzaavellino, salvatore.caddemi, ivo.calio}@unict.it

Abstract: This paper presents a study of the influence of damage intensities on the dynamic response of beams with an arbitrary number of cracks. Crack locations are considered as deterministic whereas their depths are assumed as interval uncertain but bounded variables. Explicit although approximated expressions of the main modal parameters related as a function of the crack severities are provided and the reliability of the proposed formulas is verified with the exact solutions. The methodology relies on modelling the local stiffness reduction at cracks by means of generalised functions (distributions), which allows the formulation of closed form expressions of the mode shapes and the relevant frequency equation in presence of an arbitrary number of cracks. The proposed methodology, which keeps the size of the problem constant as the number of cracks increases and avoids the need to perform re-analysis of the problem, is here adopted to assess upper and lower bounds of the response of beams with multiple cracks subjected to deterministic loads by making use of both time and frequency domain analyses. The efficacy of the proposed approach is corroborated by numerical examples on simple damaged straight beams and allows future advancements for similar analyses on damaged frames in presence of crack uncertainties.

Keywords: Dynamic response, Generalised functions, concentrated damage, Sherman-Morrison formula, interval analysis

1. Introduction

Structural deterministic analyses lead to a deterministic response. However, such an approach can be considered reliable only in presence of structures and load scenarios free of uncertainties. Nevertheless, it is widely accepted that uncertainties in the mechanical properties or in the loads acting on a structure strongly affect the expected response. This simple principle is accounted for in most of the structural codes (Eurocode 1, 1991, FEMA 356, 2000) adopted by practitioners, to assure a reasonable safety factor in the structural design and assessment.

One of the most dangerous aspects for the safety of structures is the presence of damage. The presence of impairments in a structure is often hidden and its magnitude can be mostly considered uncertain. For such a reason investigating how the structural response can be affected by a damage intensity represents a challenging issue whose solution can have a significant practical impact.

Damage can occur in structures in a diffused or concentrated pattern. Usually, especially in presence of an early detection, damage can be approximately considered concentrated. One of the most widely adopted model to simulate the presence of damage in a structure is the so called equivalent rotational hinge model (Irwin, 1957a and 1957b), according to which the presence of a damage in a straight or curved beam can be modelled by disconnecting the two sides of the beam and inserting a rotational spring whose stiffness can be related to the crack depth according to different models (Paipetis and Dimarogonas, 1986, Gudmnundson, 1983, Bilello 2001). Classical applications of structural theories follow the strategy of modelling beam-like structures according to a Euler-Bernoulli or Timoshenko theory, and require the enforcement of continuity conditions at the cracked sections; however, the main drawback is that the problem size increases with the number of cracked sections. An effective way to treat discontinuities due to concentrated cracks is to adopt a mathematical distributional model which avoids enforcing continuity conditions at the cracked sections. Accordingly, boundary conditions are required at the two ends of the beam only, irrespectively of the along-axis cracks present in the beam. Such an approach has been successfully applied to deterministic problems in the static, stability and dynamic fields, also with reference to frames and curved beams and to inverse problems as well (Caddemi et al., 2015, Palmeri and Cicirello, 2011, Yavari et al., 2000, Wang and Qiao, 2007, Cannizzaro et al, 2017, Di Lorenzo et al., 2017, Failla and Impollonia, 2012). However, it has never been applied to problems with uncertainties.

The modelling of uncertainties can be performed according to different models. One of the most complete and satisfactory widely used strategy is to adopt a stochastic model where a given parameter is described according to a classical probabilistic approach (Muscolino et al., 2000, Impollonia and Ricciardi, 2006, Falsone and Impollonia, 2002). In this case the output of the structural analysis is characterised by a probability density function. Other strategies make use of the fuzzy logic (Moens and Vandepitte, 2002) which assumes that a given value can be associated to an uncertain parameter with a certain level of 'truth' ranging from 0 to 1.

In the last years an alternative method which gained importance in the relevant literature is the interval analysis. Such a strategy is based on the assumption that an uncertain parameter can vary within a given range, without assuming any probability content; under such hypotheses the goal of the interval analysis is to infer the corresponding bounds of a response parameter (Alefeld and Herzberger, 1984, Moore et al. 2009). This methodology has a significant importance for the engineering practice since it provides a tool to explore the variability of the response due to the input parameter uncertainties and is able to provide the maximum and minimum values of the response, which have the greatest relevance for practical purposes. The interval analysis has been applied to several problems, mainly related to statics, dynamic frequency response of structures and transient analysis (Impollonia and Muscolino, 2011, Muscolino and Sofi, 2013, Xia et al., 2010). With regard to damaged structures, the interval analysis has been mainly employed for inverse problems (Campi et al., 2009). The direct problem for damaged beams with uncertain but bounded intensities, at least in the dynamic field, was treated in (Muscolino and Santoro, 2019) within the framework of a classic FEM approach.

In this paper, beam structures assumed as continuous Euler-Bernoulli models, are treated by making use of the generalized approach previously mentioned to model the presence of concentrated cracks, and introducing a model of uncertainty for the crack severities (Cannizzaro et al. 2020). The main modal parameters (frequencies, generalized masses and load terms in case of forced vibrations), are then approximated with explicit expressions, whose reliability is duly verified. Then, according to this approximated approach, the forced vibrations of cracked beams are assessed, and the bounds are computed in a simplified manner. In spite of the limited computational effort needed for the interval analysis, it is shown how the proposed approach is able to reproduce the bounds of the response computed according to an exact re-analysis (Kirsch, 2008) associated to a significant computational burden. Remarkably, the proposed procedure is able to account for non-monotonic relationship between input and output variables. Then, for forced vibration the extreme values of the response can be detected even at an internal point of the input variable interval, when convex analysis (Hu and Qiu, 2010) or other simplified procedures fail.

2. The adopted model

When a multi-cracked beam is considered, classic approaches require FEM analyses or the enforcement of continuity conditions at the cracked sections to obtain its response. To make more effective this procedure, a solution strategy of the governing equations of a multi-cracked beam, based on generalised functions, is here employed (Cannizzaro et al., 2018). The solution, proposed in explicit closed form and here briefly recalled, is unique and is provided in terms of four (boundary conditions dependent) integration constants only, and, unlike classic approaches, does not require any additional node, or continuity condition, at the cracked sections.

Let us consider an Euler-Bernoulli beam, with a dimensionless spatial abscissa x spanning from 0 to the length L and distributed mass m, in presence of multiple cracks at x_i=1,…,n, and a generic transverse load $\bar{q}(x,t)$, as shown in Figure 1.

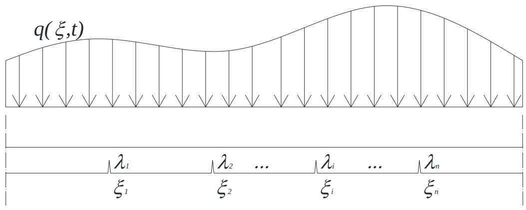


Figure 1. Damaged beam subjected to a generic time-dependent external transverse load.

By introducing the normalised abscissa ξ=x/L, the dimensionless governing equation of the vibratory motion of the deterministic beam in terms of normalized deflection function $u(\xi,t)=v(x,t)/L$ with t time variable, is written as follows:

$$u^{IV}(\xi,t)+\frac{mL^4}{E_oI_o}\ddot{u}(\xi,t)=\sum_{i=1}^{n}\Delta u'(\xi_i,t)\delta''(\xi-\xi_i)+q(\xi,t) \tag{1}$$

being $q(\xi,t)=\dfrac{\bar{q}(x,t)L^3}{E_oI_o}$. By introducing the dimensionless crack compliance $\lambda_i=\dfrac{E_oI_o}{K_iL}$ (being K_i the stiffness of the equivalent rotational spring), the following formula, which relates the unknown rotations at the cracked sections with the relevant bending moments, holds:

$$\Delta u'(\xi_i,t)=\lambda_i u''(\xi_i,t) \tag{2}$$

The deterministic eigenproperties associated to Eq.(1) have already been inferred in (Cannizzaro et al., 2018). In the following the damage intensities at the cracked sections will be considered as uncertain parameters collected in the vector $\boldsymbol{\lambda}=[\lambda_1, \lambda_2,\ldots, \lambda_i,\ldots, \lambda_n]$. Thus, the eigenproperties of the beam as well as its response will be dependent on the crack severities. Therefore, the generic p-th mode shape ϕ_p can be expressed as:

$$\phi_p(\xi;\boldsymbol{\lambda})=\sum_{j=1}^{4}C_{j,p}(\boldsymbol{\lambda})\,f_{j,p}(\xi;\boldsymbol{\lambda}) \tag{3}$$

where

$$f_{j,p}(\xi;\boldsymbol{\lambda})=h_{j,p}(\xi;\boldsymbol{\lambda})+\sum_{i=1}^{n}\bar{h}_{i,p}(\xi;\boldsymbol{\lambda})\lambda_i f''_{j,p}(\xi_i;\boldsymbol{\lambda}) \quad , \quad j=1,\ldots,4 \tag{4}$$

and

$$h_{1,p}(\xi;\boldsymbol{\lambda})=\sin\alpha_p\xi;\quad h_{2,p}(\xi;\boldsymbol{\lambda})=\cos\alpha_p\xi;\quad h_{3,p}(\xi;\boldsymbol{\lambda})=\sinh\alpha_p\xi;\quad h_{4,p}(\xi;\boldsymbol{\lambda})=\cosh\alpha_p\xi;$$
$$\bar{h}_{i,p}(\xi;\boldsymbol{\lambda})=\frac{1}{2\alpha_p}\left[\sin\alpha_p(\xi-\xi_i)+\sinh\alpha_p(\xi-\xi_i)\right]U(\xi-\xi_i) \tag{5}$$

being $\alpha_p^4(\boldsymbol{\lambda})=\omega_p^2 mL^4/EI$ is the p-th frequency parameter depending on the end condition of the beam and on its damage configuration. The functions $h_{j,p}(\xi,\boldsymbol{\lambda})$, j=1,…4, represent the solution of the homogeneous beam, while $\bar{h}_{i,p}(\xi;\boldsymbol{\lambda}),\ i=1,\dots,n,$ represent additional functions able to account for the presence of the cracks.

The response of the beam can be obtained through modal superposition, that is once the eigenvalue problem of the multi-cracked beam has been solved for the natural frequencies and modes, the displacement is given by a linear combination of the modes:

$$u(\xi,t;\boldsymbol{\lambda})=\sum_{r=1}^{\infty}\phi_p(\xi;\boldsymbol{\lambda})z_p(t)\cong\sum_{r=1}^{n_m}\phi_p(\xi;\boldsymbol{\lambda})z_p(t) \tag{6}$$

being $z_p(t)$ the p-th modal coordinate and n_m the number of considered modes. In the modal space, by exploiting the orthogonality properties of the mode shapes and considering a viscous damping, the governing equations are

$$\ddot{z}_p(t)+2\zeta_p\alpha_p^2(\boldsymbol{\lambda})\dot{z}_p(t)+\alpha_p^4(\boldsymbol{\lambda})z_p(t)=\frac{Q_p(t;\boldsymbol{\lambda})}{M_p(\boldsymbol{\lambda})}\qquad p=1,\dots,n_m \tag{7}$$

being ζ_p the modal damping associated to the generic p-th mode. The generalized modal mass and the modal load term appearing in Eq.(7) are given by the following formulas

$$M_p(\boldsymbol{\lambda})=m\int_0^1\phi_p(\xi;\boldsymbol{\lambda})\phi_p(\xi;\boldsymbol{\lambda})\,d\xi;\qquad Q_p(t;\boldsymbol{\lambda})=\int_0^1\phi_p(\xi;\boldsymbol{\lambda})q(\xi,t)\,d\xi \tag{8}$$

The integration constants appearing in Eq.(3) can be obtained by enforcing the relevant boundary conditions. Without loss of generality, here two cases corresponding to clamped-clamped and pinned-pinned conditions are elucidated in Table I.

Table I. Boundary conditions and determinantal equations

Clamped-clamped	$\phi(0;\boldsymbol{\lambda})=0;\quad \phi'(0;\boldsymbol{\lambda})=0;$ $\phi(1;\boldsymbol{\lambda})=0;\quad \phi'(1;\boldsymbol{\lambda})=0$	$\left[f_{3,p}(1;\boldsymbol{\lambda})-f_{1,p}(1;\boldsymbol{\lambda})\right]\left[f'_{4,p}(1;\boldsymbol{\lambda})-f'_{2,p}(1;\boldsymbol{\lambda})\right]+$ $-\left[f_{4,p}(1;\boldsymbol{\lambda})-f_{2,p}(1;\boldsymbol{\lambda})\right]\left[f'_{3,p}(1;\boldsymbol{\lambda})-f'_{1,p}(1;\boldsymbol{\lambda})\right]=0$
Pinned-pinned	$\phi(0;\boldsymbol{\lambda})=0;\quad \phi''(0;\boldsymbol{\lambda})=0;$ $\phi(1;\boldsymbol{\lambda})=0;\quad \phi''(1;\boldsymbol{\lambda})=0$	$f_{1,p}(1;\boldsymbol{\lambda})\,f''_{3,p}(1;\boldsymbol{\lambda})-f_{3,p}(1;\boldsymbol{\lambda})\,f''_{1,p}(1;\boldsymbol{\lambda})=0$

3. Interval uncertainty model

An explicit approximated formulation is here proposed to assess the variability of the dynamic response of a damaged beam associated to a non-deterministic parameter in the beam configuration free of any computational burden usually required by the previously addressed exact formulation. In particular, the crack severities will be considered as uncertain variables in given intervals. According to the classical interval arithmetic, the generic variable λ_i can range in the interval $\lambda_i \in \left[\underline{\lambda}_i, \overline{\lambda}_i\right]$ characterized by the midpoint value $\lambda_{o,i}$.Given the fluctuation amplitude $\Delta\lambda_i$ the following relations hold:

$$\lambda_{o,i} = \frac{1}{2}\left(\underline{\lambda}_i + \overline{\lambda}_i\right); \qquad \Delta\lambda_i = \frac{1}{2}\left(\overline{\lambda}_i - \underline{\lambda}_i\right) \tag{9}$$

where the symbols $\underline{\bullet}$ and $\overline{\bullet}$ indicate the lower and the upper bounds, respectively. A generic value of the *i*-th crack severity λ_i can be then expressed as:

$$\lambda_j = \lambda_{o,j} + \beta_j, \qquad \beta_i \in \left[-\Delta\lambda_i, \Delta\lambda_i\right] \tag{10}$$

Depending on the cracks severities, the modal parameters change consequently, and need to be assessed to compute the response of the beam. In particular, the needed parameters for the generic *p*-th mode are the frequency parameter $\alpha_p^4(\boldsymbol{\lambda})$, the load term $Q_p(t;\boldsymbol{\lambda})$, and the generalized modal mass $M_p(\boldsymbol{\lambda})$, see Eqs. (8). Since the exact computation of such parameters requires to find the roots of the characteristic equation for the frequency parameters $\alpha_p^4(\boldsymbol{\lambda})$, and the computation of integrals depending on the *p*-th mode shape, in the following an approximated but explicit approach to evaluate the mentioned modal parameters is introduced.
Let us consider the case of a generic cracked beam with uncertain severities $\boldsymbol{\lambda}$ according to the model introduced in Eqs. (9). In order to provide an approximated evaluation of the main modal parameters of the damaged beam affected by uncertainty in the damage intensities, 2*n* significant damage configurations will be considered, and the correspondence between exact and approximated computations of the frequency parameter $\alpha_p^4(\boldsymbol{\lambda})$ for such configurations will be enforced. Each of the *n* pairs of these configurations can be defined as follows:

$$\begin{aligned} \overline{\boldsymbol{\lambda}}_s &= \left[\lambda_{o,1}, \lambda_{o,2}, \ldots, \overline{\lambda}_s, \ldots \lambda_{o,n}\right] \\ \underline{\boldsymbol{\lambda}}_s &= \left[\lambda_{o,1}, \lambda_{o,2}, \ldots, \underline{\lambda}_s, \ldots \lambda_{o,n}\right] \end{aligned} \qquad s = 1, \ldots, n \tag{11}$$

The following interpolating formula, proposed in (Sherman and Morrison, 1949) for the inversion of an invertible matrix and already employed in (Impollonia, 2006), is adapted for the treated case and here proposed to provide an estimation of the interval frequency parameters as functions of the uncertain damage configuration:

$$\begin{aligned} \alpha_p^4(\boldsymbol{\lambda}) &\cong \alpha_{o,p}^4 + \sum_{i=1}^{n} \frac{\beta_i a_{p,i}}{1+\beta_i b_{p,i}} \\ M_p(\boldsymbol{\lambda}) &\cong M_{o,p} + \sum_{i=1}^{n} \frac{\beta_i c_{p,i}}{1+\beta_i d_{p,i}} \qquad p = 1, \ldots, n_m, \quad \beta_i \in \left[-\Delta\lambda_i, \Delta\lambda_i\right] \\ Q_p(t;\boldsymbol{\lambda}) &\cong Q_{o,p}(t) + \sum_{i=1}^{n} \frac{\beta_i e_{p,i}(t)}{1+\beta_i f_{p,i}(t)} \end{aligned} \tag{12}$$

being $\alpha_{o,p}^4$, $M_{o,p}$ and $Q_{o,p}(t)$ the p-th modal parameters associated to the reference distribution $\boldsymbol{\lambda}_\mathbf{o}=[\lambda_{o,1}, \lambda_{o,2},\ldots, \lambda_{o,n}]$ of damage, whereas $a_{p,i}$, $b_{p,i}$, $c_{p,i}$, $d_{p,i}$ and $e_{p,i}(t)$, $f_{p,i}(t)$ represent appropriate sets of coefficients evaluated explicitly by enforcing the correspondence between exact and approximate expression of the frequency parameters at the $2n$ considered crack configurations $\underline{\boldsymbol{\lambda}}_s, \overline{\boldsymbol{\lambda}}_s$, $s=1,\ldots,n$ previously defined in Eq. (11), which can be explicitly evaluated with the following formulas:

$$\begin{aligned}
a_{p,s} &= \frac{2}{\Delta\lambda_s}\frac{\left[\alpha_p^4\left(\overline{\boldsymbol{\lambda}}_s\right)-\alpha_{o,p}^4\right]\left[\alpha_{o,p}^4-\alpha_p^4\left(\underline{\boldsymbol{\lambda}}_s\right)\right]}{\alpha_p^4\left(\overline{\boldsymbol{\lambda}}_s\right)-\alpha_p^4\left(\underline{\boldsymbol{\lambda}}_s\right)}; \quad b_{p,s} = \frac{1}{\Delta\lambda_s}\frac{2\alpha_{o,p}^4-\alpha_p^4\left(\underline{\boldsymbol{\lambda}}_s\right)-\alpha_p^4\left(\overline{\boldsymbol{\lambda}}_s\right)}{\alpha_p^4\left(\overline{\boldsymbol{\lambda}}_s\right)-\alpha_p^4\left(\underline{\boldsymbol{\lambda}}_s\right)} \\
c_{p,s} &= \frac{2}{\Delta\lambda_s}\frac{\left[M_p\left(\overline{\boldsymbol{\lambda}}_s\right)-M_{o,p}\right]\left[M_{o,p}-M_p\left(\underline{\boldsymbol{\lambda}}_s\right)\right]}{M_p\left(\overline{\boldsymbol{\lambda}}_s\right)-M_p\left(\underline{\boldsymbol{\lambda}}_s\right)}; \qquad d_{p,s} = \frac{1}{\Delta\lambda_s}\frac{2M_{o,p}-M_p\left(\underline{\boldsymbol{\lambda}}_s\right)-M_p\left(\overline{\boldsymbol{\lambda}}_s\right)}{M_p\left(\overline{\boldsymbol{\lambda}}_s\right)-M_p\left(\underline{\boldsymbol{\lambda}}_s\right)} \\
e_{p,s}(t) &= \frac{2}{\Delta\lambda_s}\frac{\left[Q_p\left(t;\overline{\boldsymbol{\lambda}}_s\right)-Q_{o,p}(t)\right]\left[Q_{o,p}(t)-Q_p\left(t;\underline{\boldsymbol{\lambda}}_s\right)\right]}{Q_p\left(t;\overline{\boldsymbol{\lambda}}_s\right)-Q_p\left(t;\underline{\boldsymbol{\lambda}}_s\right)}; \; f_{p,s}(t) = \frac{1}{\Delta\lambda_s}\frac{2Q_{o,p}(t)-Q_p\left(t;\underline{\boldsymbol{\lambda}}_s\right)-Q_p\left(t;\overline{\boldsymbol{\lambda}}_s\right)}{Q_p\left(t;\overline{\boldsymbol{\lambda}}_s\right)-Q_p\left(t;\underline{\boldsymbol{\lambda}}_s\right)}
\end{aligned} \tag{13}$$

with p=1,…,n_m and s=1,…,n. In the following Figure 2, a validation in terms of accuracy of the proposed procedure is presented. A clamped-clamped beam, with a crack located at ξ_1=0.2 associated to a reference severity $\lambda_{o,1}$=0.15 in investigated; the considered amplitude for the damage severity interval is $\Delta\lambda_1$=0.15. A concentrated transversal unit step load is considered at the abscissa ξ_o=0.7. In particular, in Figures 2b, 2d, 2f the comparisons between exact and approximated parameters are shown for the first five frequencies, that is frequency parameters (Figure 2b), generalized modal mass (Figure 2d) and load term (Figure 2f); all the terms are normalized by the corresponding reference value (see Table 2). In Figure 2c, 2e, 2g the corresponding relative errors are reported. The continuous lines correspond to the exact properties, whereas the dashed lines are relative to the approximated values computed according to Eqs. (12).

Table II. Reference values of the main modal properties for the beam reported in Figure 2a

p	$\alpha_{o,p}^4$	$M_{o,p}$	$Q_{o,p}$
1	498.34	1.074	1.143
2	3257.52	0.850	-1.311
3	12315.67	1.108	1.051
4	35523.70	1.769	0.349
5	86618.26	1.645	-1.769

The goal of the present formulation is to explicitly assess the bounds of a generic response term $r(\xi_r,t;\boldsymbol{\lambda})$ at a desired abscissa ξ_r and, as a particular case, the corresponding bounds within given intervals of the uncertain parameters. Consistently with the model adopted for the uncertain parameters in Eq. (10) the minimum and maximum bounds of the response will be indicated in the following as $\underline{r}\left(\xi_r,t;\boldsymbol{\lambda}\right)$ and $\overline{r}\left(\xi_r,t;\boldsymbol{\lambda}\right)$, respectively. In order to obtain such bounds of the response, a standard procedure would require to compute $r(\xi_r,t;\boldsymbol{\lambda})$ according to the exact procedure for a finite number n_λ of configurations associated to the uncertain parameters $\boldsymbol{\lambda}$ in the given intervals, for example at given equally spaced steps. Such a procedure requires n_λ

computations of the fundamental frequencies, as well as solving the integrals in Eqs. (8); in addition, unless $n_\lambda \to \infty$ the bounds of the response can be evaluated only in an approximate way.

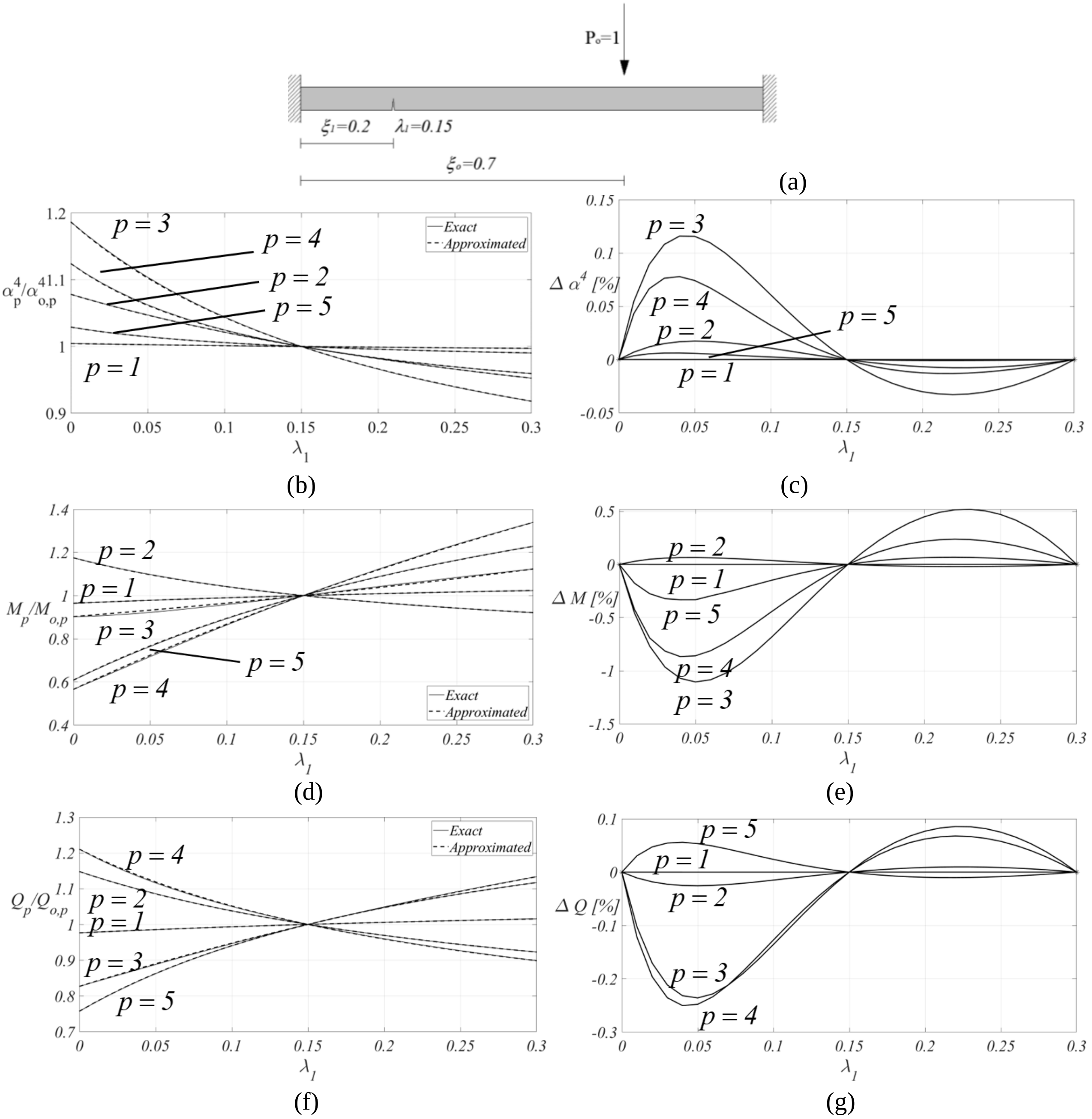


Figure 2. Comparison between exact and approximated modal properties for a (a) single cracked beam: (b) frequency parameter and (c) relative error, (d) generalized modal mass and (e) relative error, (f) load term and (g) relative error

An alternative procedure is to express the response of the beam according to the approximated but explicit formulation in terms of fundamental frequencies $\alpha_p^4(\boldsymbol{\lambda})$, generalized masses $M_p(\boldsymbol{\lambda})$ and load terms $Q_p(t;\boldsymbol{\lambda})$. The generic response of the beam can be expressed as:

$$r\left(\xi_r,t;\boldsymbol{\lambda}\right)=\sum_{p=1}^{n_m}\phi_p^k\left(\xi_r;\boldsymbol{\lambda}\right)z_p\left(t\right) \tag{14}$$

being k the order of the derivative associated to desired response $r(\xi_r,t;\boldsymbol{\lambda})$. Depending on the value assumed by k, the response in terms of deflection (k = 0), rotation (k = 1), bending moment (k = 2), shear force (k = 3) can be inferred. The mode shapes and their derivatives can be obtained in an approximated way by replacing in Eqs. (3)-(5) the approximated frequency parameter computed according to Eqs. (12); the approximated modal coordinates, considering the entire response or only the steady-state response, will require the approximated computation of the modal generalized mass and of the load term according to the scheme proposed in Eqs. (12).

Under these hypotheses, which require the exact solution of the problem at $2n$+1 given configurations of the damage pattern, namely $\boldsymbol{\lambda}_\mathbf{o};\overline{\boldsymbol{\lambda}}_s,\underline{\boldsymbol{\lambda}}_s, s=1,...,n$, the dependency of the dynamic response on the damage intensities can be explicitly provided; thus, an approximated but explicit assessment of the dynamic response variability is obtained. A further resulting important advantage of the proposed procedure lies in the possibility of inferring the bounds of a certain response parameter; to this purpose, from the symbolic response expression, which is function of the damage intensities, the bounds of the response, associated to uncertain but bounded intervals of the damage severities, can be simply obtained by solving two problems of maximum/minimum of constrained nonlinear functions.

4. Numerical applications

The explicit approach described in the previous sections is here applied with reference to two meaningful applications for the forced vibration analysis of damaged beams. The presented applications concern with the search for the bounds of response parameters within given intervals of the crack depths both in terms of response spectra and time histories.

In the first subsection, with reference to the steady-state response only, spectra of the response for a single cracked beam subjected to pulsating loads will be proposed and compared again with the results obtained through the deterministic response associated to the scanning of the damage configurations at given steps. Then, in second subsection, the bounds of the response of a double cracked beam will be assessed and compared with the response computed by scanning the damage intensity interval and applying the exact deterministic analysis for each value. In this case both the transient and the steady-state responses will be accounted for.

It will be shown how the bounds of the response are not always associated to the bounds of the intervals of the crack depth parameter, thus proving that the procedure is more robust than other approaches.

4.1. Interval frequency-response curves

In the dynamics of structures, a useful tool to summarize the behaviour of a structure subjected to frequency-dependent loads is the frequency-response curve. Basically, given a pulsating load with frequency ω applied to a structure, a frequency-response curve collects the magnification factor of the steady-state response with respect to the corresponding static response (Chopra, 2001). In an undamped deterministic analysis the frequency-response curve tends to diverge when ω assumes a value corresponding to each of the natural frequencies of the system (resonance phenomenon); even in presence of damping the peaks of the responses (not divergent) can be encountered in correspondence of the damped natural frequencies. When an uncertainty of the crack severity in a beam is considered, the natural frequencies can change significantly, as already shown in Figure 2; therefore, it is expected that an interval frequency-response band will show that the bounds of the response can be associated to values of the damage severities which do not coincide with the bounds of the considered interval.

To this purpose an application is here proposed with reference to the beam reported in Figure 2a and subjected to a concentrated pulsating load at the midspan of the beam. The reference output parameter is the transversal displacement at the abscissa corresponding to the midspan of the beam. The first five frequencies are accounted for and the re-analysis is associated to 31 analyses (considering a damage step size $\Delta\tilde{\lambda}_1$ =0.01). In Figure 3 the results are shown for the case of undamped system. The steady-state response $u_{ss}(\xi_r)/\ u_{st}(\xi_r)$ at the desired abscissa divided by the corresponding response of the beam when subjected to the static load and then normalized by the corresponding response of the beam associated to the reference damage configuration $u_{ss,o}(\xi_r)$ versus the normalized frequency parameter of the pulsating load $\eta = \alpha_\omega^2/\alpha_{o,1}^2$ is reported. The exact responses associated to three significant damage configurations are reported in the same plots, namely the reference configuration (i.e. λ_1=0.15), and those associated to the healthy beam (i.e. λ_1=0) and to the most severe configuration (i.e. λ_1=0.3) with solid, dashed, and dashdot lines, respectively, Figure 3a. Such responses are not able to envelope correctly the grey area (see for example the second and the third resonance frequencies) and the classical vertex method does not seem suitable to assess the bounds of the response when an uncertainty in the damage intensities is introduced. In Figure 3b the bounds of the response obtained with the proposed approach are superimposed to the re-analysis envelope.

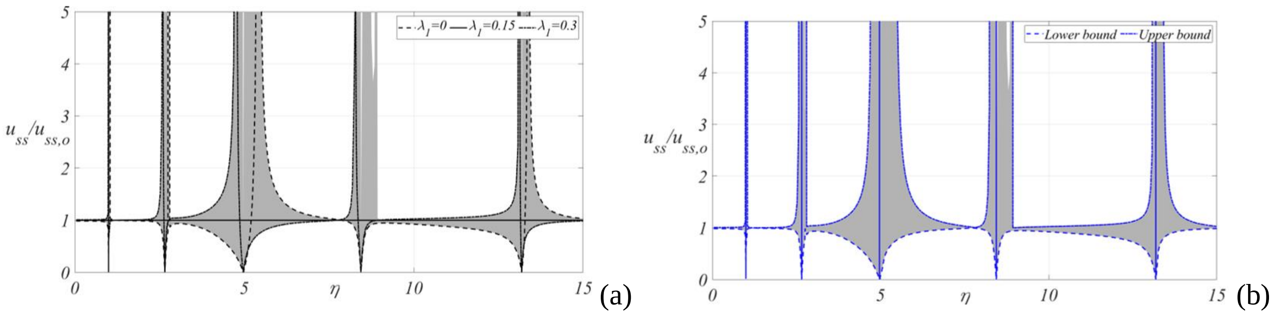


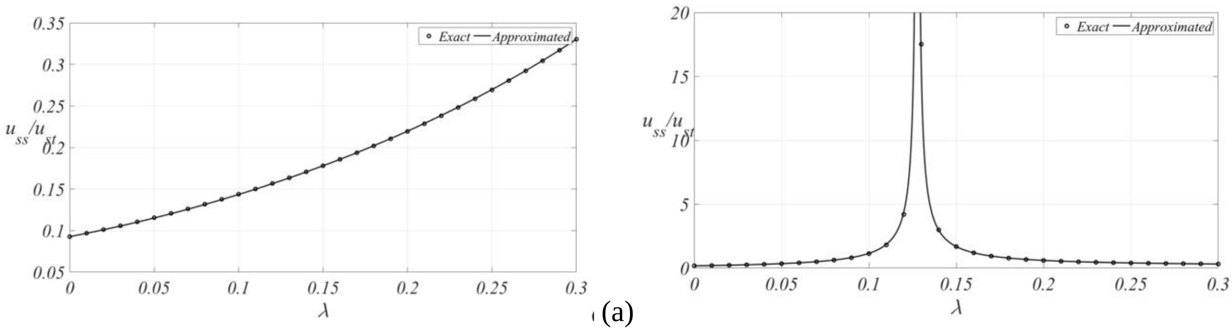
Figure 3. Normalized frequency-response interval spectrum for the undamped beam reported in Figure 2a: comparison between the exact results associated to a re-analysis (grey band) and (a) exact analyses associated to boundary intensities of damage and (b) response bounds obtained with the proposed approach

Three main remarks can be made observing the latter results:

i) the proposed approximated explicit approach is able to bind the envelope obtained with the re-analysis for the whole considered range of the load frequency;

ii) the envelope obtained through re-analysis, being based on a discrete evaluation of the response for a finite number of values of the damage severity of the crack, is itself an approximated way to assess the actual envelope of the dynamic response; theoretically, the re-analysis is able to regain the exact response variability only when an infinite number of values of the damage severities is considered;

iii) in this regard, the proposed approximated solution appears to be even more reliable than a re-analysis associated to a finite number of damage configurations (see for example the range $8 \leq \eta \leq 9$).
The dependency of the steady state response on the crack intensity is further investigated considering two values of the pulsation, namely η=4.5 (Figure 4a) and η=5.0 (Figure 4b). In the first case a monotonic trend of the monitored response parameter is encountered as the damage intensity increases; in the second case a non-monotonic trend is observed, and the response tends to diverge for a damage intensity $\lambda_f \approx 0.1278$, thus showing that a vertex method is not suitable for the treated case.



(a) (b)

Figure 4. Crack intensity vs steady state response for the beam reported in Figure 2a for specific frequency ratios: (a) η =4.5 and (b) η =5.0

4.2. INTERVAL TIME-DOMAIN RESPONSE

The effectiveness of the proposed approximated solution is assessed in this subsection by comparing the bounds of the response computed according to a scanning exact procedure with those obtained through the procedure proposed in this paper.
Aiming at highlighting the advantages offered by the proposed procedure when multiple cracks are considered, in Figure 5 the results relative to a pinned-pinned double cracked beam as reported in Figure 5a are presented; the beam is subjected to a concentrated pulsating load acting on the midpoint of the beam, $q(\xi,t)=P_o\delta(\xi-\xi_L)\sin\omega t$ (P_o=1, ξ_L =0.5, ω=10 rad/s). The two cracks are located at ξ_1=0.2 and ξ_2=0.7, respectively, and are both characterized by the reference intensity $\lambda_1= \lambda_2$=0.05. The approximated modal parameters are computed considering a fluctuation amplitude $\Delta\lambda_1=\Delta\lambda_2$=0.05. Again, the grey area represents the envelope of the exact responses computed scanning the damage intensities at evenly spaced steps $\Delta\tilde{\lambda}_1 = \Delta\tilde{\lambda}_2 = 0.01$; therefore, 121 exact analyses were performed, that is the computational burden increases with the second power of the number of cracks. On the other hand, the results relative to the interval analysis obtained with the proposed approach require the computation of the exact modal parameters for five configurations, that is $\boldsymbol{\lambda}_o = [0.05, 0.05]$, $\overline{\boldsymbol{\lambda}}_1 = [0.1, 0.05]$, $\underline{\boldsymbol{\lambda}}_1 = [0, 0.05]$, $\overline{\boldsymbol{\lambda}}_2 = [0.05, 0.1]$, $\underline{\boldsymbol{\lambda}}_2 = [0.05, 0]$, and then solving two maximum/minimum conditioned problems for each time step. The monitored output parameter is again the transversal displacement at the midpoint of the beam, and the first three frequencies are considered with an associated damping equal to 0.05 for all the modes.
In Figure 5b and 5c the grey area represents the ensemble of the exact responses computed through re-analysis. For convenience, the exact responses for the reference damage, and those associated to the lowest

and the highest crack severities (i.e. $\lambda_1 = \lambda_2 = 0$ and $\lambda_1 = \lambda_2 = 0.1$) are reported in Figure 5b with continuous, dashed, and dashdot lines, respectively. It may be noticed that the bounds of the responses identified by the contour of the grey area cannot be identified by simply running the three mentioned exact analyses, and even running a larger number of analyses for given values of damage severities (in this case 121 exact analyses were run), the correct bounds of the response can be identified only in an approximate way. Theoretically the correct bounds of the response would require a very large number of analyses. Conversely, in Figure 5c the bounds of the response are computed according to the proposed procedure. The relevant lower and upper bounds are reported with the blue dashed and dashdot lines, respectively. The approach here proposed is able to effectively assess the bounds of the response for all the time history analysis, and it is also associated to a significantly lower computational burden with respect to a re-analysis approach.

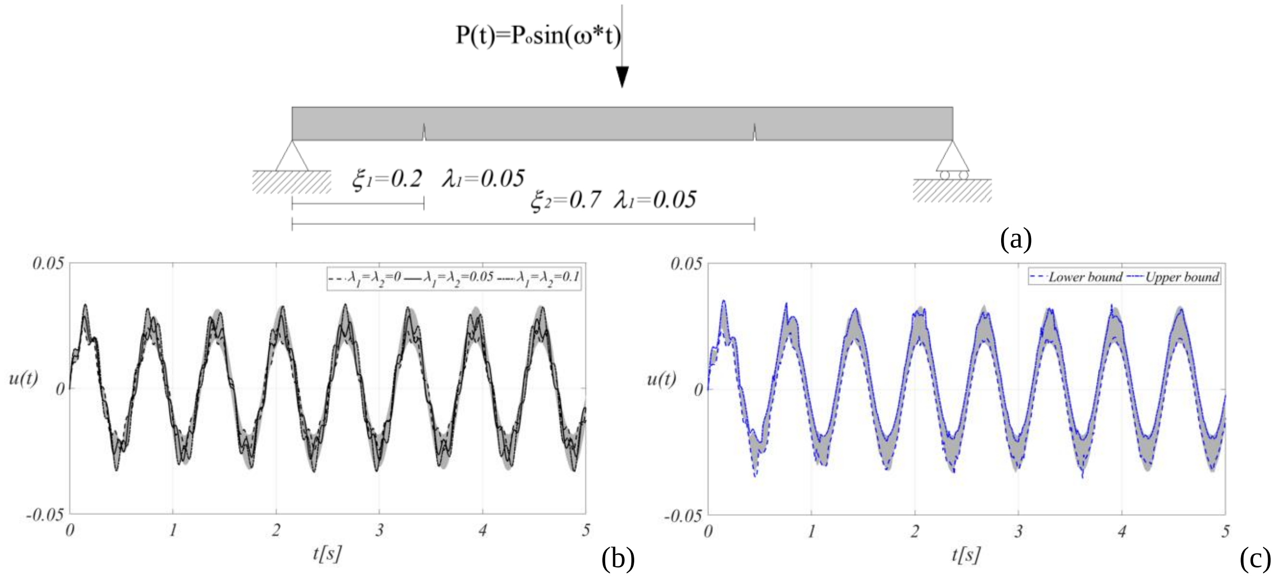


Figure 5. Interval time-domain response for the (a) beam subjected to a concentrated pulsating load: comparison between the exact results associated to a re-analysis (grey band) and (b) exact analyses associated to boundary intensities of damage and (c) response bounds obtained with the proposed approach

5. Conclusions

In this paper uncertainties in the dynamic response of damaged beams are treated within the framework of the interval analysis; in particular, damage severities are considered uncertain. To this purpose, a model previously introduced for the deterministic dynamic analysis of multi-cracked beams, able to avoid the enforcing of continuity conditions at the cracked sections, is employed. Here, taking advantage of an approximated approach to express main modal parameters (frequency parameters, generalized modal mass and load term), the response to the forced vibration of cracked beam is inferred explicitly, thus making possible to easily infer the bounds of the response associated to given intervals of the damage severities. The accuracy of the approximated approach for the evaluation of the modal parameters is compared with reference solutions; then, the method is applied to single and multi-cracked beams, both with reference to transient time history analyses and to the steady-state response. The proposed approach is much faster with respect to a classic scanning approach and brought to light that for the forced vibrations of damaged structures the bounds of the response are not necessarily associated to the most and less severe configurations, but might involve intermediate damage levels.

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